4 Roots and Powers

BUILDING ON

- determining the square root of a positive rational number
- applying the exponent laws for powers with integral bases and whole number exponents

BIG IDEAS

- Any number that can be written as the fraction $\frac{m}{n}$, $n \neq 0$, where m and n are integers, is rational.
- Exponents can be used to represent roots and reciprocals of rational numbers.
- The exponent laws can be extended to include powers with rational and variable bases, and rational exponents.

NEW VOCABULARY

irrational number
real number
entire radical
mixed radical

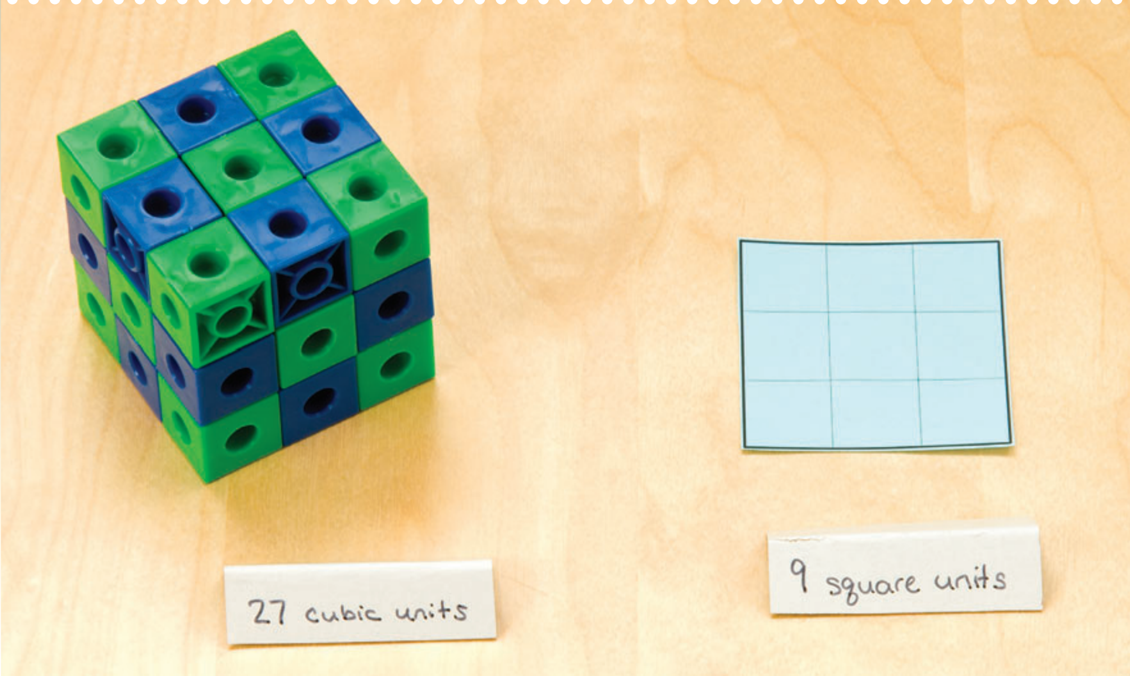
YUKON QUEST *This is a sled dog race from Whitehorse, Yukon, to Fairbanks, Alaska.*

MATH LAB

4.1 Estimating Roots

LESSON FOCUS

Explore decimal representations of different roots of numbers.



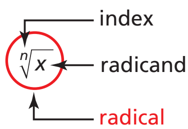
Make Connections

Since $3^2 = 9$, 3 is a square root of 9.

We write: $3 = \sqrt{9}$

Since $3^3 = 27$, 3 is the cube root of 27.

We write: $3 = \sqrt[3]{27}$



Since $3^4 = 81$, 3 is a fourth root of 81.

We write: $3 = \sqrt[4]{81}$

How would you write 5 as a square root? A cube root? A fourth root?

Construct Understanding

TRY THIS

Work with a partner.

You will need a calculator to check your estimates.

A. Write the two consecutive perfect squares closest to 20. Estimate the value of $\sqrt{20}$. Square your estimate. Use this value to revise your estimate. Keep revising your estimate until the square of the estimate is within 1 decimal place of 20.

B. Write the two consecutive perfect cubes closest to 20. Estimate the value of $\sqrt[3]{20}$. Cube your estimate. Use this value to revise your estimate. Keep revising your estimate until the cube of the estimate is within 1 decimal place of 20.

C. Write the two consecutive perfect fourth powers closest to 20. Use a strategy similar to that in Steps A and B to estimate a value for $\sqrt[4]{20}$.

D. Copy and complete this table. Use the strategies from Steps A to C to determine the value of each radical.

Radical	Value	Is the Value Exact or Approximate?
$\sqrt{16}$	4	Exact
$\sqrt{27}$	5.1962	Approximate
$\sqrt{\frac{16}{81}}$	$\frac{4}{9}$ or $0.\overline{4}$	Exact
$\sqrt{0.64}$		
$\sqrt[3]{16}$		
$\sqrt[3]{27}$		
$\sqrt[3]{\frac{16}{81}}$		
$\sqrt[3]{0.64}$		
$\sqrt[3]{-0.64}$		
$\sqrt[4]{16}$		
$\sqrt[4]{27}$		
$\sqrt[4]{\frac{16}{81}}$		
$\sqrt[4]{0.64}$		

Choose 3 different radicals.
Extend then complete the table for these radicals.

E. How can you tell if the value of a radical is a rational number? What strategies can you use to determine the value of the radical?

F. How can you tell if the value of a radical is *not* a rational number? What strategies can you use to estimate the value of the radical?

Assess Your Understanding

1. **a)** Give 4 examples of radicals. Use a different index for each radical.
b) Identify the radicand and index for each radical.
c) Explain the meaning of the index of each radical.

2. Evaluate each radical. Justify your answer.

a) $\sqrt{36}$ **b)** $\sqrt[3]{8}$ **c)** $\sqrt[4]{10\,000}$ **d)** $\sqrt[5]{-32}$

e) $\sqrt[3]{\dfrac{27}{125}}$ **f)** $\sqrt{2.25}$ **g)** $\sqrt[3]{0.125}$ **h)** $\sqrt[4]{625}$

3. Estimate the value of each radical to 1 decimal place.
What strategy did you use?

a) $\sqrt{8}$ **b)** $\sqrt[3]{9}$ **c)** $\sqrt[4]{10}$ **d)** $\sqrt{13}$

e) $\sqrt[3]{15}$ **f)** $\sqrt[4]{17}$ **g)** $\sqrt{19}$ **h)** $\sqrt[3]{20}$

4. **a)** What happens when you attempt to determine the square root of a number such as –4? Explain the result.
b) For which other radical indices do you get the same result with a negative radicand, as in part a?
c) When a radicand is negative:
i) Which types of radicals can be evaluated or estimated?
ii) Which types of radicals cannot be evaluated or estimated?

5. For each number below, write an equivalent form as:
i) a square root **ii)** a cube root **iii)** a fourth root

a) 2 **b)** 3 **c)** 4 **d)** 10 **e)** 0.9 **f)** 0.2

6. Choose values of n and x so that $\sqrt[n]{x}$ is:
a) a whole number **b)** a negative integer
c) a rational number **d)** an approximate decimal
Verify your answers.

4.2 Irrational Numbers

LESSON FOCUS
Identify and order irrational numbers.

The room below the rotunda in the Manitoba Legislative Building is the Pool of the Black Star. It has a circular floor.

Make Connections

The formulas for the area and circumference of a circle involve π, which is not a rational number because it cannot be written as a quotient of integers.

What other numbers are not rational?

Construct Understanding

TRY THIS

Work with a partner.

These are rational numbers.	These are not rational numbers.
$\sqrt{100}$ $\sqrt{0.25}$ $\sqrt[3]{8}$ 0.5	$\sqrt{0.24}$ $\sqrt[3]{9}$ $\sqrt{2}$
$\frac{5}{6}$ $\sqrt{\frac{9}{64}}$ 0.8^2 $\sqrt[5]{-32}$	$\sqrt{\frac{1}{3}}$ $\sqrt[4]{12}$

A. How are radicals that are rational numbers different from radicals that are not rational numbers?

B. Which of these radicals are rational numbers?
Which are not rational numbers? How do you know?

$\sqrt{1.44}$, $\sqrt{\frac{64}{81}}$, $\sqrt[3]{-27}$, $\sqrt{\frac{4}{5}}$, $\sqrt{5}$

C. Write 3 other radicals that are rational numbers.
Why are they rational?

D. Write 3 other radicals that are not rational numbers.
Why are they not rational?

Radicals that are square roots of perfect squares, cube roots of perfect cubes, and so on are rational numbers. Rational numbers have decimal representations that either terminate or repeat.

Irrational Numbers

An **irrational number** *cannot* be written in the form $\frac{m}{n}$, where m and n are integers, $n \neq 0$. The decimal representation of an irrational number neither terminates nor repeats.

When an irrational number is written as a radical, the radical is the *exact value* of the irrational number; for example, $\sqrt{2}$ and $\sqrt[3]{-50}$. We can use the square root and cube root keys on a calculator to determine *approximate values* of these irrational numbers.

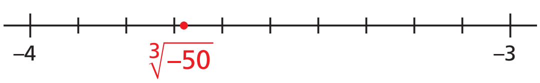
There are other irrational numbers besides radicals; for example, π.

We can approximate the location of an irrational number on a number line. If we do not have a calculator, we use perfect powers to estimate the value. For example, to locate $\sqrt[3]{-50}$ on a number line,
we know that $\sqrt[3]{-27} = -3$ and $\sqrt[3]{-64} = -4$.

Guess: $\sqrt[3]{-50} \doteq -3.6$ Test: $(-3.6)^3 = -46.656$

Guess: $\sqrt[3]{-50} \doteq -3.7$ Test: $(-3.7)^3 = -50.653$

This is close enough to represent on a number line.



Since $(-3.7)^3 = -50.653$, then $\sqrt[3]{-50}$ is slightly greater than -3.7, so mark a point to the right of -3.7 on the number line.

Example 1 Classifying Numbers

Tell whether each number is rational or irrational. Explain how you know.

a) $-\frac{3}{5}$ b) $\sqrt{14}$ c) $\sqrt[3]{\frac{8}{27}}$

SOLUTION

a) $-\frac{3}{5}$ is rational since it is written as a quotient of integers.

Its decimal form is -0.6, which terminates.

b) $\sqrt{14}$ is irrational since 14 is not a perfect square.

The decimal form of $\sqrt{14}$ neither repeats nor terminates.

c) $\sqrt[3]{\frac{8}{27}}$ is rational since $\frac{8}{27}$ is a perfect cube.

$\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$ or $0.\overline{6}$, which is a repeating decimal

CHECK YOUR UNDERSTANDING

1. Tell whether each number is rational or irrational. Explain how you know.

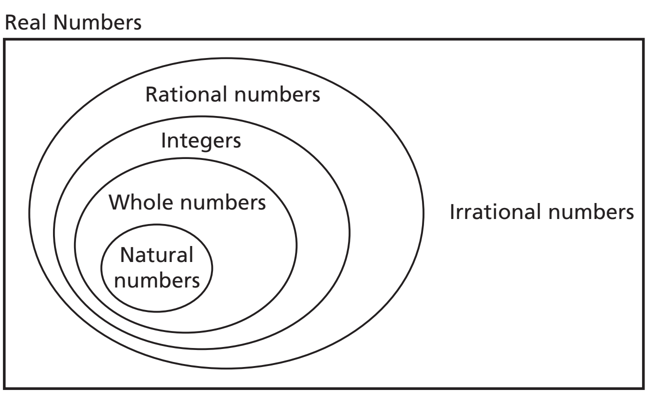
a) $\sqrt{\frac{49}{16}}$ b) $\sqrt[3]{-30}$

c) 1.21

[Answers: a) rational
b) irrational c) rational]

Together, the rational numbers and irrational numbers form the set of **real numbers**.

This diagram shows how these number systems are related.



Example 2 Ordering Irrational Numbers on a Number Line

Use a number line to order these numbers from least to greatest.

$\sqrt[3]{13}$, $\sqrt{18}$, $\sqrt{9}$, $\sqrt[4]{27}$, $\sqrt[3]{-5}$

(Solution continues.)

SOLUTION

13 is between the perfect cubes 8 and 27, and is closer to 8.

$$\begin{array}{ccc} \sqrt[3]{8} & \sqrt[3]{13} & \sqrt[3]{27} \\ \downarrow & \downarrow & \downarrow \\ 2 & ? & 3 \end{array}$$

Use a calculator.

$\sqrt[3]{13} = 2.3513\ldots$

3 x√(13)
2.351334688

18 is between the perfect squares 16 and 25, and is closer to 16.

$$\begin{array}{ccc} \sqrt{16} & \sqrt{18} & \sqrt{25} \\ \downarrow & \downarrow & \downarrow \\ 4 & ? & 5 \end{array}$$

Use a calculator.

$\sqrt{18} = 4.2426\ldots$

√(18)
4.242640687

$\sqrt{9} = 3$

27 is between the perfect fourth powers 16 and 81, and is closer to 16.

$$\begin{array}{ccc} \sqrt[4]{16} & \sqrt[4]{27} & \sqrt[4]{81} \\ \downarrow & \downarrow & \downarrow \\ 2 & ? & 3 \end{array}$$

Use a calculator.

$\sqrt[4]{27} = 2.2795\ldots$

4 x√(27)
2.279507057

-5 is between the perfect cubes -1 and -8, and is closer to -8.

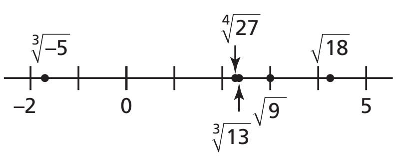
$$\begin{array}{ccc} \sqrt[3]{-1} & \sqrt[3]{-5} & \sqrt[3]{-8} \\ \downarrow & \downarrow & \downarrow \\ -1 & ? & -2 \end{array}$$

Use a calculator.

$\sqrt[3]{-5} = -1.7099\ldots$

3 x√(-5)
-1.709975947

Mark each number on a number line.



From least to greatest: $\sqrt[3]{-5}, \sqrt[4]{27}, \sqrt[3]{13}, \sqrt{9}, \sqrt{18}$

CHECK YOUR UNDERSTANDING

2. Use a number line to order these numbers from least to greatest.

$\sqrt{2},\ \sqrt[3]{-2},\ \sqrt[3]{6},\ \sqrt{11},\ \sqrt[4]{30}$

[Answer: $\sqrt[3]{-2}, \sqrt{2}, \sqrt[3]{6}, \sqrt[4]{30}, \sqrt{11}$]

How can you order a set of irrational numbers if you do not have a calculator?

Discuss the Ideas

1. How do you determine whether a radical represents a rational or an irrational number? Use examples to explain.

2. How can you determine whether the decimal form of a radical represents its exact value?

Exercises

A

3. Tell whether each number is rational or irrational.
   a) $\sqrt{12}$ b) $\sqrt[4]{16}$
   c) $\sqrt[3]{-100}$ d) $\sqrt{\frac{4}{9}}$
   e) $\sqrt{1.25}$ f) 1.25

4. Classify each number below as:
   a) a natural number b) an integer
   c) a rational number d) an irrational number

   $\frac{4}{3}$, $0.3\overline{4}$, –5, $\sqrt[4]{9}$, -2.1538, $\sqrt[3]{27}$, 7

B

5. a) Why are $\sqrt{49}$ and $\sqrt[4]{16}$ rational numbers?
   b) Why are $\sqrt{21}$ and $\sqrt[3]{36}$ irrational numbers?

6. Look at this calculator screen.

   √(150)
   12.24744871

   a) Is the number 12.247 448 71 rational or irrational? Explain.
   b) Is the number $\sqrt{150}$ rational or irrational? Explain.

7. a) Sketch a diagram to represent the set of rational numbers and the set of irrational numbers.
   b) Write each number that follows in the correct set.
   $\frac{1}{2}$, $-\sqrt{3}$, $\sqrt{4}$, $\sqrt[4]{5}$, $-\frac{7}{6}$, $\sqrt[3]{8}$, 10.12, $-13.\overline{4}$, $\sqrt{0.15}$, $\sqrt{0.16}$, 17

8. For which numbers will the cube root be irrational? Use 2 different strategies to justify your answers.
   a) 8 b) 64 c) 30 d) 300

9. Sketch a number line for each irrational number and label its approximate location. Explain your reasoning.
   a) $\sqrt{5}$ b) $\sqrt[3]{12}$ c) $\sqrt[4]{25}$ d) $\sqrt[3]{-12}$

10. Use a number line to order the irrational numbers in each set from greatest to least.
    a) $\sqrt[3]{70}$, $\sqrt{50}$, $\sqrt[4]{100}$, $\sqrt[3]{400}$
    b) $\sqrt{89}$, $\sqrt[4]{250}$, $\sqrt[3]{-150}$, $\sqrt[3]{150}$

11. Use a number line to order these numbers from least to greatest. How can you verify your answer?
    $\sqrt{40}$, $\sqrt[3]{500}$, $\sqrt{98}$, $\sqrt[3]{98}$, $\sqrt{75}$, $\sqrt[3]{300}$

12. Use a number line to order these numbers from least to greatest. Identify which numbers are irrational and which are rational.
    $\frac{-14}{5}$, $\frac{123}{99}$, -2, $\sqrt[3]{-10}$, $\sqrt{4}$

13. How do you use irrational numbers when you calculate the length of the hypotenuse of a right triangle with legs 5 cm and 3 cm?

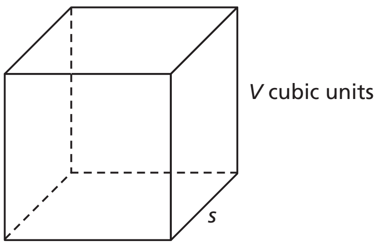
14. a) Which of the following statements are true? Explain your reasoning.
    i) All natural numbers are integers.
    ii) All integers are rational numbers.
    iii) All whole numbers are natural numbers.
    iv) All irrational numbers are roots.
    v) Some rational numbers are natural numbers.
    b) For each statement in part a that is false, provide examples to explain why.

15. Write a number that is:
    a) a rational number but not an integer
    b) a whole number but not a natural number
    c) an irrational number

16. **a)** Create a diagram to show how these number systems are related: irrational numbers; rational numbers; integers; whole numbers; and natural numbers.
Write each number below in your diagram.
$\frac{3}{5}$, 4.91919, 16, $\sqrt[3]{-64}$, $\sqrt{60}$, $\sqrt{9}$, -7, 0

b) Choose 3 more numbers that would be appropriate for each number system in your diagram. Include these numbers.

17. This diagram shows a cube with volume V cubic units and edge length s units.



Provide a value of V for which s is:

a) irrational **b)** rational

18. The *golden rectangle* appears in art and architecture. It has the property that the ratio of its length to width is $\frac{1+\sqrt{5}}{2}$ to 1. The shape of the front of the Parthenon, in Greece, is a golden rectangle.

a) Use a calculator. Write the value of $\frac{1+\sqrt{5}}{2}$ to the nearest tenth.

b) Use the number from part a as the length in inches of a rectangle. Draw a golden rectangle.

c) Measure other rectangles in your classroom. Do any of these rectangles approximate a golden rectangle? Justify your answer.

19. The ratio $\frac{1+\sqrt{5}}{2}$: 1 is called the *golden ratio.* Use the dimensions of the Great Pyramid of Giza, from Chapter 1, page 26. Show that the ratio of its base side length to its height approximates the golden ratio.

20. Determine whether the perimeter of each square is a rational number or an irrational number. Justify your answer.

a) a square with area 40 cm^2

b) a square with area 81 m^2

C

21. Suppose you know that $\sqrt[n]{\frac{a}{b}}$ is a rational number and that a and b have no common factors. What can you say about the prime factorizations of a and b?

22. For each description, sketch and label a right triangle or explain why it is not possible to create a triangle.

a) All sides have rational number lengths.

b) Exactly 2 sides have rational number lengths.

c) Exactly 1 side has a rational number length.

d) No sides have rational number lengths.

23. **a)** Can the square root of a rational number be irrational? Show your reasoning.

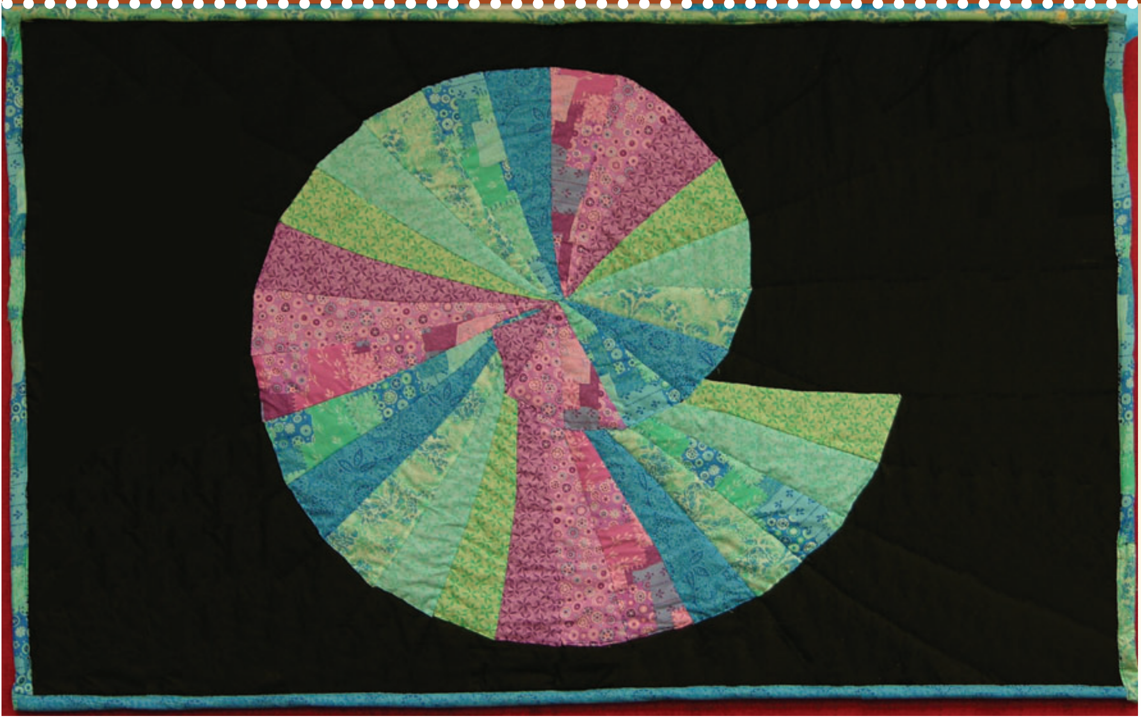
b) Can the square root of an irrational number be rational? Show your reasoning.

24. Describe strategies to generate numbers with the property that their square roots, cube roots, and fourth roots are all rational numbers. Support your strategies with examples.

Reflect

Describe strategies you can use to determine if a radical represents a rational number or an irrational number.

4.3 Mixed and Entire Radicals



LESSON FOCUS

Express an entire radical as a mixed radical, and vice versa.

This quilt represents a Pythagorean spiral. The smallest triangle is a right isosceles triangle with legs 1 unit long.

Make Connections

We can name the fraction $\frac{3}{12}$ in many different ways:

$\frac{1}{4}$ $\quad \frac{5}{20}$ $\quad \frac{30}{120}$ $\quad \frac{100}{400}$

How do you show that each fraction is equivalent to $\frac{3}{12}$?
Why is $\frac{1}{4}$ the simplest form of $\frac{3}{12}$?

Construct Understanding

TRY THIS

Work with a partner.

You will need 1-cm grid paper and a calculator.

A. On grid paper, draw an isosceles right triangle with legs 1 cm long.
Write the length of the hypotenuse as a radical.
Label the lengths of the sides on the triangle.

B. Draw an isosceles right triangle with legs 2 cm long.
Write the length of the hypotenuse as a radical.
Label the lengths of the sides on the triangle.

C. Explain why the triangle in Step B is an enlargement of the triangle in Step A.
What is the scale factor of the enlargement?
How is the length of the hypotenuse in the larger triangle related to the corresponding length in the smaller triangle?

D. Draw isosceles right triangles with legs: 3 cm long; 4 cm long; and 5 cm long. For each triangle, write the length of the hypotenuse in 2 different ways.

E. Describe any relationships in the lengths of the sides of the triangles. Which form of the radical makes the relationships easier to see?

Just as with fractions, equivalent expressions for any number have the same value.

- $\sqrt{16 \cdot 9}$ is equivalent to $\sqrt{16} \cdot \sqrt{9}$ because:

  $\sqrt{16 \cdot 9} = \sqrt{144}$ $= 12$ and $\sqrt{16} \cdot \sqrt{9} = 4 \cdot 3$ $= 12$

- Similarly, $\sqrt[3]{8 \cdot 27}$ is equivalent to $\sqrt[3]{8} \cdot \sqrt[3]{27}$ because:

  $\sqrt[3]{8 \cdot 27} = \sqrt[3]{216}$ $= 6$ and $\sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3$ $= 6$

Multiplication Property of Radicals

$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$,
where n is a natural number, and a and b are real numbers

We can use this property to simplify square roots and cube roots that are not perfect squares or perfect cubes, but have factors that are perfect squares or perfect cubes.
For example, the factors of 24 are: 1, 2, 3, 4, 6, 8, 12, and 24.

- We can simplify $\sqrt{24}$ because 24 has a perfect square factor of 4.
  Rewrite 24 as the product of two factors, one of which is 4.

$$\begin{aligned}\sqrt{24} &= \sqrt{4 \cdot 6} \\ &= \sqrt{4} \cdot \sqrt{6} \\ &= 2 \cdot \sqrt{6} \\ &= 2\sqrt{6}\end{aligned}$$

We read $2\sqrt{6}$ as "2 root 6."

- Similarly, we can simplify $\sqrt[3]{24}$ because 24 has a perfect cube factor of 8. Rewrite 24 as the product of two factors, one of which is 8.

$$\begin{aligned}\sqrt[3]{24} &= \sqrt[3]{8 \cdot 3}\\ &= \sqrt[3]{8} \cdot \sqrt[3]{3}\\ &= 2 \cdot \sqrt[3]{3}\\ &= 2\sqrt[3]{3}\end{aligned}$$

We read $2\sqrt[3]{3}$ as "2 cube root 3."

- However, we cannot simplify $\sqrt[4]{24}$ because 24 has no factors (other than 1) that can be written as a fourth power.

We can also use prime factorization to simplify a radical.

Example 1 Simplifying Radicals Using Prime Factorization

Simplify each radical.

a) $\sqrt{80}$ **b)** $\sqrt[3]{144}$ **c)** $\sqrt[4]{162}$

SOLUTION

Write each radical as a product of prime factors, then simplify.

a)
$$\begin{aligned}\sqrt{80} &= \sqrt{8 \cdot 10}\\ &= \sqrt{2 \cdot 2 \cdot 2 \cdot 5 \cdot 2}\\ &= \sqrt{(2 \cdot 2) \cdot (2 \cdot 2) \cdot 5}\\ &= \sqrt{2 \cdot 2} \cdot \sqrt{2 \cdot 2} \cdot \sqrt{5}\\ &= 2 \cdot 2 \cdot \sqrt{5}\\ &= 4\sqrt{5}\end{aligned}$$

Since $\sqrt{80}$ is a square root, look for factors that appear twice.

b)
$$\begin{aligned}\sqrt[3]{144} &= \sqrt[3]{12 \cdot 12}\\ &= \sqrt[3]{2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 \cdot 3}\\ &= \sqrt[3]{(2 \cdot 2 \cdot 2) \cdot 2 \cdot 3 \cdot 3}\\ &= \sqrt[3]{2 \cdot 2 \cdot 2} \cdot \sqrt[3]{2 \cdot 3 \cdot 3}\\ &= 2 \cdot \sqrt[3]{2 \cdot 3 \cdot 3}\\ &= 2\sqrt[3]{18}\end{aligned}$$

Since $\sqrt[3]{144}$ is a cube root, look for factors that appear 3 times.

c)
$$\begin{aligned}\sqrt[4]{162} &= \sqrt[4]{81 \cdot 2}\\ &= \sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3 \cdot 2}\\ &= \sqrt[4]{(3 \cdot 3 \cdot 3 \cdot 3) \cdot 2}\\ &= \sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3} \cdot \sqrt[4]{2}\\ &= 3\sqrt[4]{2}\end{aligned}$$

Since $\sqrt[4]{162}$ is a fourth root, look for factors that appear 4 times.

CHECK YOUR UNDERSTANDING

1. Simplify each radical.

a) $\sqrt{63}$

b) $\sqrt[3]{108}$

c) $\sqrt[4]{128}$

[Answers: a) $3\sqrt{7}$ b) $3\sqrt[3]{4}$ c) $2\sqrt[4]{8}$]

Some numbers, such as 200, have more than one perfect square factor.
The factors of 200 are: 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200
Since 4, 25, and 100 are perfect squares, we can simplify $\sqrt{200}$ in these ways.

$$\begin{aligned}\sqrt{200} &= \sqrt{4 \cdot 50} \\ &= \sqrt{4} \cdot \sqrt{50} \\ &= 2\sqrt{50}\end{aligned} \qquad \begin{aligned}\sqrt{200} &= \sqrt{25 \cdot 8} \\ &= \sqrt{25} \cdot \sqrt{8} \\ &= 5\sqrt{8}\end{aligned} \qquad \begin{aligned}\sqrt{200} &= \sqrt{100 \cdot 2} \\ &= \sqrt{100} \cdot \sqrt{2} \\ &= 10\sqrt{2}\end{aligned}$$

$10\sqrt{2}$ is in simplest form because the radical contains no perfect square factors other than 1.

To write a radical of index n in simplest form, we write the radicand as a product of 2 factors, one of which is the greatest perfect nth power.

Example 2 Writing Radicals in Simplest Form

Write each radical in simplest form, if possible.

a) $\sqrt[3]{40}$ **b)** $\sqrt{26}$ **c)** $\sqrt[4]{32}$

SOLUTION

Look for perfect nth factors, where n is the index of the radical.

a) The factors of 40 are: 1, 2, 4, 5, 8, 10, 20, 40
The greatest perfect cube is $8 = 2 \cdot 2 \cdot 2$,
so write 40 as $8 \cdot 5$.

$$\begin{aligned}\sqrt[3]{40} &= \sqrt[3]{8 \cdot 5} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{5} \\ &= 2 \cdot \sqrt[3]{5} \\ &= 2\sqrt[3]{5}\end{aligned}$$

b) The factors of 26 are: 1, 2, 13, 26
There are no perfect square factors other than 1.
So, $\sqrt{26}$ cannot be simplified.

c) The factors of 32 are: 1, 2, 4, 8, 16, 32
The greatest perfect fourth power is $16 = 2 \cdot 2 \cdot 2 \cdot 2$,
so write 32 as $16 \cdot 2$.

$$\begin{aligned}\sqrt[4]{32} &= \sqrt[4]{16 \cdot 2} \\ &= \sqrt[4]{16} \cdot \sqrt[4]{2} \\ &= 2 \cdot \sqrt[4]{2} \\ &= 2\sqrt[4]{2}\end{aligned}$$

CHECK YOUR UNDERSTANDING

2. Write each radical in simplest form, if possible.

a) $\sqrt{30}$

b) $\sqrt[3]{32}$

c) $\sqrt[4]{48}$

[Answers: a) cannot be simplified b) $2\sqrt[3]{4}$ c) $2\sqrt[4]{3}$]

Radicals of the form $\sqrt[n]{x}$ such as $\sqrt{80}$, $\sqrt[3]{144}$, and $\sqrt[4]{162}$ are **entire radicals.**

Radicals of the form $a\sqrt[n]{x}$ such as $4\sqrt{5}$, $2\sqrt[3]{18}$, and $3\sqrt[4]{2}$ are **mixed radicals.** Entire radicals were rewritten as mixed radicals in *Examples 1* and *2*.

Any number can be written as the square root of its square; for example, $2 = \sqrt{2 \cdot 2}$, $3 = \sqrt{3 \cdot 3}$, $4 = \sqrt{4 \cdot 4}$, and so on. Similarly, any number can be written as the cube root of its cube, or the fourth root of its perfect fourth power. We use this strategy to write a mixed radical as an entire radical.

Example 3 Writing Mixed Radicals as Entire Radicals

Write each mixed radical as an entire radical.

a) $4\sqrt{3}$ **b)** $3\sqrt[3]{2}$ **c)** $2\sqrt[5]{2}$

SOLUTION

a) Write 4 as: $\sqrt{4 \cdot 4} = \sqrt{16}$

$$4\sqrt{3} = \sqrt{16} \cdot \sqrt{3}$$

Use the Multiplication Property of Radicals.

$$= \sqrt{16 \cdot 3}$$

$$= \sqrt{48}$$

b) Write 3 as: $\sqrt[3]{3 \cdot 3 \cdot 3} = \sqrt[3]{27}$

$$3\sqrt[3]{2} = \sqrt[3]{27} \cdot \sqrt[3]{2}$$

$$= \sqrt[3]{27 \cdot 2}$$

$$= \sqrt[3]{54}$$

c) Write 2 as: $\sqrt[5]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \sqrt[5]{32}$

$$2\sqrt[5]{2} = \sqrt[5]{32} \cdot \sqrt[5]{2}$$

$$= \sqrt[5]{32 \cdot 2}$$

$$= \sqrt[5]{64}$$

CHECK YOUR UNDERSTANDING

3. Write each mixed radical as an entire radical.

a) $7\sqrt{3}$

b) $2\sqrt[3]{4}$

c) $2\sqrt[5]{3}$

[Answers: a) $\sqrt{147}$ b) $\sqrt[3]{32}$ c) $\sqrt[5]{96}$]

How would rewriting mixed radicals as entire radicals help you to order a set of mixed radicals with the same index?

Discuss the Ideas

1. How can you determine if an entire radical can be written as a mixed radical?

2. Suppose an entire radical can be simplified. How do you use the Multiplication Property of Radicals to write it in simplest form?

Exercises

A

3. List all the perfect squares up to 400, and their square roots.

4. Write each radical in simplest form.

a) $\sqrt{8}$ b) $\sqrt{12}$ c) $\sqrt{32}$ d) $\sqrt{50}$

e) $\sqrt{18}$ f) $\sqrt{27}$ g) $\sqrt{48}$ h) $\sqrt{75}$

5. Write each mixed radical as an entire radical.

a) $5\sqrt{2}$ b) $6\sqrt{2}$ c) $7\sqrt{2}$ d) $8\sqrt{2}$

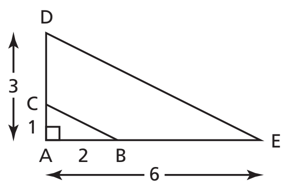
e) $5\sqrt{3}$ f) $6\sqrt{3}$ g) $7\sqrt{3}$ h) $8\sqrt{3}$

6. a) List all the perfect cubes up to 1000, and their cube roots.

b) List all the perfect fourth powers up to 1000, and their fourth roots.

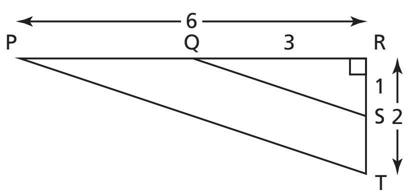
B

7. a) Use the diagram to explain why $\sqrt{45} = 3\sqrt{5}$.



b) Use algebra to verify that $\sqrt{45} = 3\sqrt{5}$.

8. a) Use the diagram to explain why $\sqrt{40} = 2\sqrt{10}$.



b) Use algebra to verify that $\sqrt{40} = 2\sqrt{10}$.

9. Explain why rewriting $\sqrt{50}$ as $\sqrt{25} \cdot \sqrt{2}$ helps you simplify $\sqrt{50}$, but rewriting $\sqrt{50}$ as $\sqrt{10} \cdot \sqrt{5}$ does not.

10. Write each radical in simplest form, if possible.

a) $\sqrt{90}$ b) $\sqrt{73}$ c) $\sqrt{108}$

d) $\sqrt{600}$ e) $\sqrt{54}$ f) $\sqrt{91}$

g) $\sqrt{28}$ h) $\sqrt{33}$ i) $\sqrt{112}$

11. Write each radical in simplest form, if possible.

a) $\sqrt[3]{16}$ b) $\sqrt[3]{81}$

c) $\sqrt[3]{256}$ d) $\sqrt[3]{128}$

e) $\sqrt[3]{60}$ f) $\sqrt[3]{192}$

g) $\sqrt[3]{135}$ h) $\sqrt[3]{100}$

i) $\sqrt[3]{500}$ j) $\sqrt[3]{375}$

12. Write each mixed radical as an entire radical.

a) $3\sqrt{2}$ b) $4\sqrt{2}$

c) $6\sqrt{5}$ d) $5\sqrt{6}$

e) $7\sqrt{7}$ f) $2\sqrt[3]{2}$

g) $3\sqrt[3]{3}$ h) $4\sqrt[3]{3}$

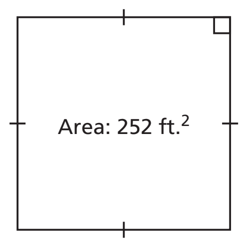
i) $5\sqrt[3]{2}$ j) $2\sqrt[3]{9}$

13. a) Can every mixed radical be expressed as an entire radical?

b) Can every entire radical be expressed as a mixed radical?

Give examples to support your answers.

14. Express the side length of this square as a radical in simplest form.



15. A cube has a volume of 200 cm^3. Write the edge length of the cube as a radical in simplest form.

16. A square has an area of 54 square inches. Determine the perimeter of the square. Write the answer as a radical in simplest form.

17. Write each radical in simplest form.

a) $\sqrt[4]{48}$ b) $\sqrt[4]{405}$

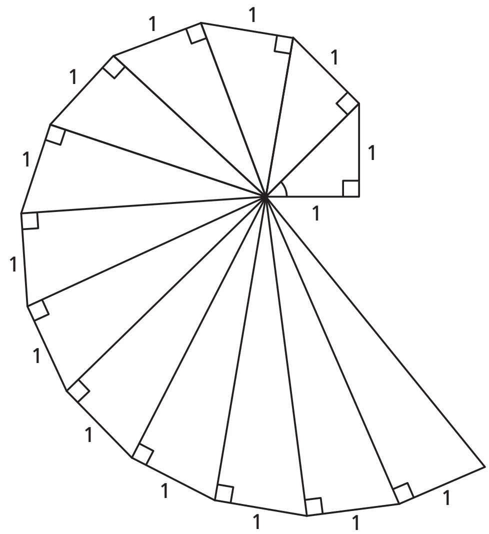
c) $\sqrt[4]{1250}$ d) $\sqrt[4]{176}$

18. Write each mixed radical as an entire radical.

a) $6\sqrt[4]{3}$ b) $7\sqrt[4]{2}$

c) $3\sqrt[5]{4}$ d) $4\sqrt[5]{3}$

19. The quilt on page 213 is made from right triangles. In Chapter 2, page 77, you determined the tangents of the angles at the centre of the spiral. The first triangle is a right isosceles triangle with legs 1 unit long. The hypotenuse of this triangle is one leg of the second triangle, with its other leg 1 unit long. This pattern continues.



a) Calculate the length of the hypotenuse of each triangle. Write each length as an entire radical.

b) i) What pattern do you see in the lengths?

ii) Use this pattern to predict the length of the hypotenuse of the 50th triangle.

iii) How many of the first 100 triangles have hypotenuse lengths that can be written as mixed radicals? Justify your answer.

20. Here is a student's solution for writing $8\sqrt[3]{2}$ as an entire radical.

$$8\sqrt[3]{2} = 8 \cdot \sqrt[3]{2}$$
$$= \sqrt[3]{2} \cdot \sqrt[3]{2}$$
$$= \sqrt[3]{2 \cdot 2}$$
$$= \sqrt[3]{4}$$

Identify an error the student made, then write the correct solution.

21. A student simplified $\sqrt{96}$ as shown:

$$\sqrt{96} = \sqrt{4} \cdot \sqrt{48}$$
$$= 2 \cdot \sqrt{48}$$
$$= 2 \cdot \sqrt{8} \cdot \sqrt{6}$$
$$= 2 \cdot 4 \cdot \sqrt{6}$$
$$= 8\sqrt{6}$$

Identify the errors the student made, then write a correct solution.

22. Arrange in order from greatest to least. What strategy did you use each time?

a) $9\sqrt{2}, 2\sqrt{6}, 8\sqrt{3}, 4\sqrt{5}, 6\sqrt{2}$

b) $4\sqrt{7}, 8\sqrt{3}, 2\sqrt{13}, 6\sqrt{5}$

c) $7\sqrt{3}, 9\sqrt{2}, 5\sqrt{6}, \sqrt{103}, 3\sqrt{17}$

23. Simplify the radicals in each list. What patterns do you see in the results? Write the next 2 radicals in each list.

a) $\sqrt{4}$, $\sqrt{400}$, $\sqrt{40\,000}$

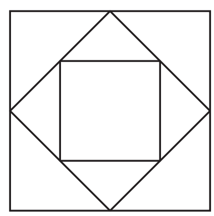
b) $\sqrt[3]{27}$, $\sqrt[3]{27\,000}$, $\sqrt[3]{27\,000\,000}$

c) $\sqrt{8}$, $\sqrt{800}$, $\sqrt{80\,000}$

d) $\sqrt[3]{24}$, $\sqrt[3]{24\,000}$, $\sqrt[3]{24\,000\,000}$

C

24. The largest square in this diagram has side length 8 cm. Calculate the side length and area of each of the two smaller squares. Write the radicals in simplest form.



25. Given that $\sqrt{2} \doteq 1.4142$, determine a decimal approximation for each radical, without using a calculator.

a) i) $\sqrt{200}$ ii) $\sqrt{20\,000}$

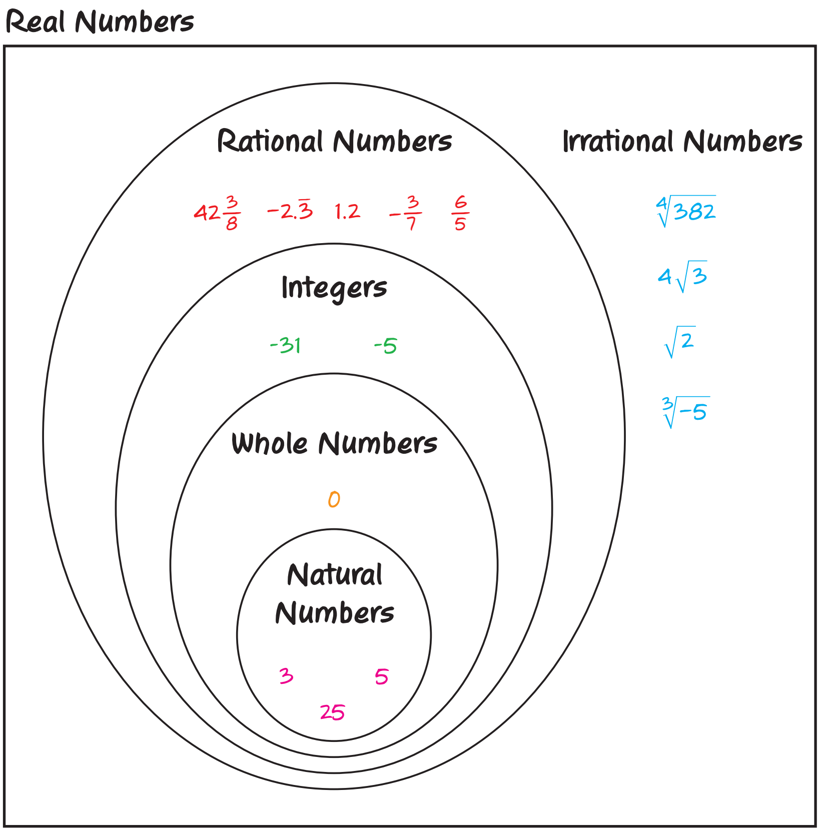
b) i) $\sqrt{8}$ ii) $\sqrt{18}$ iii) $\sqrt{32}$ iv) $\sqrt{50}$

Reflect

How do you use the index of a radical when you simplify a radical, and when you write a mixed radical as an entire radical? Use examples to support your explanation.

CHECKPOINT 1

Connections



Concept Development

- **In Lesson 4.1**
  – You applied what you know about **square roots** to explore **decimal approximations of cube roots and fourth roots.**
  – You determined that **some radicals** can be represented as **rational numbers** and other radicals cannot.

- **In Lesson 4.2**
  – You defined **irrational numbers**, and represented these numbers and **rational numbers as the set of real numbers.**
  – You **identified** conditions for which a **radical** has a **rational number value**, and **estimated** the values of **radicals that are irrational.**

- **In Lesson 4.3**
  – You defined **mixed radicals** and **entire radicals**, and used factoring to simplify radicals.

Assess Your Understanding

4.1

1. Evaluate each radical. How did you use the index of the radical in your work?

a) $\sqrt{81}$ **b)** $\sqrt[3]{-125}$ **c)** $\sqrt[4]{256}$ **d)** $\sqrt[5]{243}$

2. Estimate the value of each radical to 2 decimal places.
How can you do this without using the root keys on a calculator?

a) $\sqrt{10}$ **b)** $\sqrt[3]{15}$ **c)** $\sqrt[4]{9}$ **d)** $\sqrt[5]{23}$

3. Does the decimal representation of $\sqrt[4]{60}$ repeat, terminate, or neither?
Justify your answer.

4.2

4. Tell whether each number is rational or irrational. Justify your answers.

a) $\sqrt{11}$ **b)** $\sqrt[3]{16}$ **c)** $\sqrt[3]{-16}$ **d)** $\sqrt{121}$ **e)** $\sqrt{\frac{121}{16}}$ **f)** $\sqrt{12.1}$

5. For each irrational number, sketch a number line and label its approximate location. Describe your strategies.

a) $\sqrt{19}$ **b)** $\sqrt[3]{-20}$ **c)** $\sqrt[4]{30}$ **d)** $\sqrt[3]{36}$

6. a) Draw a diagram to illustrate the real number system.
Write the numbers below in the appropriate places on your diagram.

i) $3\frac{1}{3}$ **ii)** -42 **iii)** 4.5 **iv)** $-4.\overline{5}$

v) 0 **vi)** 14 **vii)** $\sqrt{7}$ **viii)** π

b) Choose 1 more number for each section of your diagram, where possible. Write each number in the correct place on your diagram.

7. a) Sketch a number line and mark each number on it.

i) $\sqrt{32}$ **ii)** $\sqrt[3]{72}$ **iii)** $\sqrt[4]{100}$ **iv)** $\sqrt[3]{50}$ **v)** $\sqrt{65}$ **vi)** $\sqrt[4]{60}$

b) Order the numbers in part a from greatest to least.

8. Sketch a square. Label its area so that:

a) The perimeter of the square is a rational number.

b) The perimeter of the square is an irrational number.

4.3

9. Write each radical in simplest form, if possible.

a) $\sqrt{45}$ **b)** $\sqrt[3]{96}$ **c)** $\sqrt{17}$ **d)** $\sqrt[4]{48}$ **e)** $\sqrt[3]{80}$ **f)** $\sqrt[4]{50}$

10. Choose one radical from question 9 that can be simplified.
Write a set of instructions for simplifying the radical.

11. Rewrite each mixed radical as an entire radical.

a) $3\sqrt{7}$ **b)** $2\sqrt[3]{4}$ **c)** $7\sqrt{3}$ **d)** $2\sqrt[4]{12}$ **e)** $3\sqrt[3]{10}$ **f)** $6\sqrt{11}$

4.4 Fractional Exponents and Radicals

LESSON FOCUS

Relate rational exponents and radicals.

Make Connections

Coffee, tea, and hot chocolate contain caffeine. The expression $100(0.87)^{\frac{1}{2}}$ represents the percent of caffeine left in your body $\frac{1}{2}$ h after you drink a caffeine beverage.

Given that $0.87^1 = 0.87$ and $0.87^0 = 1$, how can you estimate a value for $0.87^{\frac{1}{2}}$?

Construct Understanding

TRY THIS

Work with a partner.

A. Copy then complete each table. Use a calculator to complete the second column.

x	$x^{\frac{1}{2}}$
1	$1^{\frac{1}{2}} =$
4	$4^{\frac{1}{2}} =$
9	
16	
25	

x	$x^{\frac{1}{3}}$
1	
8	
27	
64	
125	

Continue the pattern. Write the next 3 lines in each table.

B. For each table:

- What do you notice about the numbers in the first column? Compare the numbers in the first and second columns. What conclusions can you make?
- What do you think the exponent $\frac{1}{2}$ means? Confirm your prediction by trying other examples on a calculator.
- What do you think the exponent $\frac{1}{3}$ means? Confirm your prediction by trying other examples on a calculator.

C. What do you think $a^{\frac{1}{4}}$ and $a^{\frac{1}{5}}$ mean?
Use a calculator to test your predictions for different values of a.

D. What does $a^{\frac{1}{n}}$ mean? Explain your reasoning.

In grade 9, you learned that for powers with integral bases and whole number exponents:
$a^m \cdot a^n = a^{m+n}$

To multiply powers with the same base, add the exponents.

We can extend this law to powers with fractional exponents with numerator 1:

$$5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} \quad \text{and} \quad \sqrt{5} \cdot \sqrt{5} = \sqrt{25}$$
$$= 5^1 \qquad\qquad = 5$$
$$= 5$$

$5^{\frac{1}{2}}$ and $\sqrt{5}$ are equivalent expressions; that is, $5^{\frac{1}{2}} = \sqrt{5}$.

$$\text{Similarly, } 5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} = 5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \quad \text{and} \quad \sqrt[3]{5} \cdot \sqrt[3]{5} \cdot \sqrt[3]{5} = \sqrt[3]{125}$$
$$= 5^1 \qquad\qquad = 5$$
$$= 5$$

$5^{\frac{1}{3}} = \sqrt[3]{5}$

These examples indicate that:

- Raising a number to the exponent $\frac{1}{2}$ is equivalent to taking the square root of the number.
- Raising a number to the exponent $\frac{1}{3}$ is equivalent to taking the cube root of the number, and so on.

Powers with Rational Exponents with Numerator 1

When n is a natural number and x is a rational number, $x^{\frac{1}{n}} = \sqrt[n]{x}$

Example 1 Evaluating Powers of the Form $a^{\frac{1}{n}}$

Evaluate each power without using a calculator.

a) $27^{\frac{1}{3}}$ b) $0.49^{\frac{1}{2}}$ c) $(-64)^{\frac{1}{3}}$ d) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$

SOLUTION

The denominator of the exponent is the index of the radical.

a) $27^{\frac{1}{3}} = \sqrt[3]{27}$
$= 3$

b) $0.49^{\frac{1}{2}} = \sqrt{0.49}$
$= 0.7$

c) $(-64)^{\frac{1}{3}} = \sqrt[3]{-64}$
$= -4$

d) $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}}$
$= \frac{2}{3}$

CHECK YOUR UNDERSTANDING

1. Evaluate each power without using a calculator.

a) $1000^{\frac{1}{3}}$ b) $0.25^{\frac{1}{2}}$

c) $(-8)^{\frac{1}{3}}$ d) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

[Answers: a) 10 b) 0.5 c) −2 d) $\frac{2}{3}$]

How could you check your answers?

A fraction can be written as a terminating or repeating decimal, so we can interpret powers with decimal exponents; for example, $0.2 = \frac{1}{5}$, so $32^{0.2} = 32^{\frac{1}{5}}$.

We can evaluate $32^{\frac{1}{5}}$ and $32^{0.2}$ on a calculator to show that both expressions have the same value.

```
32^(1/5)
                2
```

```
32^0.2
                2
```

Why do we use brackets to evaluate when the exponent is a fraction, but not when the exponent is a decimal?

To give meaning to a power such as $8^{\frac{2}{3}}$, we extend the exponent law $(a^m)^n = a^{mn}$ so that it applies when m and n are rational numbers.

We write the exponent $\frac{2}{3}$ as $\frac{1}{3} \cdot 2$, or as $2 \cdot \frac{1}{3}$.

So, $8^{\frac{2}{3}} = 8^{\frac{1}{3} \cdot 2}$
$= \left(8^{\frac{1}{3}}\right)^2$
$= \left(\sqrt[3]{8}\right)^2$

or

$8^{\frac{2}{3}} = 8^{2 \cdot \frac{1}{3}}$
$= (8^2)^{\frac{1}{3}}$
$= \sqrt[3]{8^2}$

Take the cube root of 8, then square the result.

So, $8^{\frac{2}{3}} = 2^2$
$= 4$

Square 8, then take the cube root of the result.

$8^{\frac{2}{3}} = \sqrt[3]{64}$
$= 4$

These examples illustrate that the numerator of a fractional exponent represents a power and the denominator represents a root. The root and power can be evaluated in any order.

Powers with Rational Exponents

When m and n are natural numbers, and x is a rational number,

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m \quad \text{and} \quad x^{\frac{m}{n}} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

Example 2 Rewriting Powers in Radical and Exponent Form

a) Write $40^{\frac{2}{3}}$ in radical form in 2 ways.

b) Write $\sqrt{3^5}$ and $\left(\sqrt[3]{25}\right)^2$ in exponent form.

SOLUTION

a) Use $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$.

$$40^{\frac{2}{3}} = \left(\sqrt[3]{40}\right)^2 \text{ or } \sqrt[3]{40^2}$$

b) Use $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

$$\sqrt{3^5} = 3^{\frac{5}{2}}$$

The index of the radical is 2.

Use $\left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$.

$$\left(\sqrt[3]{25}\right)^2 = 25^{\frac{2}{3}}$$

CHECK YOUR UNDERSTANDING

2. a) Write $26^{\frac{2}{5}}$ in radical form in 2 ways.

b) Write $\sqrt{6^5}$ and $\left(\sqrt[4]{19}\right)^3$ in exponent form.

[Answers: a) $\left(\sqrt[5]{26}\right)^2$ or $\sqrt[5]{26^2}$

b) $6^{\frac{5}{2}}$, $19^{\frac{3}{4}}$]

Example 3 Evaluating Powers with Rational Exponents and Rational Bases

Evaluate.

a) $0.04^{\frac{3}{2}}$

b) $27^{\frac{4}{3}}$

c) $(-32)^{0.4}$

d) $1.8^{1.4}$

SOLUTION

a)
$$0.04^{\frac{3}{2}} = \left(0.04^{\frac{1}{2}}\right)^3 = \left(\sqrt{0.04}\right)^3 = 0.2^3 = 0.008$$

b)
$$27^{\frac{4}{3}} = \left(27^{\frac{1}{3}}\right)^4 = \left(\sqrt[3]{27}\right)^4 = 3^4 = 81$$

(Solution continues.)

CHECK YOUR UNDERSTANDING

3. Evaluate.

a) $0.01^{\frac{3}{2}}$

b) $(-27)^{\frac{4}{3}}$

c) $81^{\frac{3}{4}}$

d) $0.75^{1.2}$

[Answers: a) 0.001 b) 81
c) 27 d) 0.7080…]

c) The exponent $0.4 = \frac{4}{10}$ or $\frac{2}{5}$

So, $(-32)^{0.4} = (-32)^{\frac{2}{5}}$

$= \left[(-32)^{\frac{1}{5}}\right]^2$

$= \left(\sqrt[5]{-32}\right)^2$

$= (-2)^2$

$= 4$

d) $1.8^{1.4}$

Use a calculator.

```
1.8^1.4
        2.277096874
```

$1.8^{1.4} = 2.2770\ldots$

The powers in parts a to c were evaluated by taking the root first. Evaluate each power by raising the base to the exponent first. Which strategy is more efficient? Justify your answer.

Example 4 Applying Rational Exponents

Biologists use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass, b kilograms, of a mammal with body mass m kilograms. Estimate the brain mass of each animal.

a) a husky with a body mass of 27 kg

b) a polar bear with a body mass of 200 kg

SOLUTION

Use the formula $b = 0.01m^{\frac{2}{3}}$.

a) Substitute: $m = 27$

$b = 0.01(27)^{\frac{2}{3}}$

$b = 0.01\left(\sqrt[3]{27}\right)^2$

$b = 0.01(3)^2$

$b = 0.01(9)$

$b = 0.09$

Use the order of operations. Evaluate the power first.

The brain mass of the husky is approximately 0.09 kg.

b) Substitute: $m = 200$

$b = 0.01(200)^{\frac{2}{3}}$

Use a calculator.

```
0.01(200)^(2/3)
        0.341995189
```

The brain mass of the polar bear is approximately 0.34 kg.

CHECK YOUR UNDERSTANDING

4. Use the formula $b = 0.01m^{\frac{2}{3}}$ to estimate the brain mass of each animal.

a) a moose with a body mass of 512 kg

b) a cat with a body mass of 5 kg

[Answers: a) approximately 0.64 kg b) approximately 0.03 kg]

Why did we use mental math to evaluate part a, but a calculator to evaluate part b?

Discuss the Ideas

1. When a is a rational number and n is a natural number, what does $a^{\frac{1}{n}}$ represent?

2. When a is a rational number and m and n are natural numbers, what does $a^{\frac{m}{n}}$ represent?

Exercises

A

3. Evaluate each power without using a calculator.
   a) $16^{\frac{1}{2}}$ b) $36^{\frac{1}{2}}$ c) $64^{\frac{1}{3}}$
   d) $32^{\frac{1}{5}}$ e) $(-27)^{\frac{1}{3}}$ f) $(-1000)^{\frac{1}{3}}$

4. Evaluate each power without using a calculator.
   a) $100^{0.5}$ b) $81^{0.25}$
   c) $1024^{0.2}$ d) $(-32)^{0.2}$

5. Write each power as a radical.
   a) $36^{\frac{1}{3}}$ b) $48^{\frac{1}{2}}$ c) $(-30)^{\frac{1}{5}}$

6. Write each radical as a power.
   a) $\sqrt{39}$ b) $\sqrt[4]{90}$
   c) $\sqrt[3]{29}$ d) $\sqrt[5]{100}$

7. Evaluate each power without using a calculator.
   a) 8^0 b) $8^{\frac{1}{3}}$ c) $8^{\frac{2}{3}}$
   d) $8^{\frac{3}{3}}$ e) $8^{\frac{4}{3}}$ f) $8^{\frac{5}{3}}$

B

8. Write each power as a radical.
   a) $4^{\frac{2}{3}}$ b) $(-10)^{\frac{3}{5}}$ c) $2.3^{\frac{3}{2}}$

9. A cube has a volume of 350 cm^3. Write the edge length of the cube as a radical and as a power.

10. Write each power as a radical.
    a) $48^{\frac{2}{3}}$ b) $(-1.8)^{\frac{5}{3}}$ c) $\left(\frac{3}{8}\right)^{2.5}$
    d) $0.75^{0.75}$ e) $\left(-\frac{5}{9}\right)^{\frac{2}{5}}$ f) $1.25^{1.5}$

11. Write each radical as a power.
    a) $\sqrt{3.8^3}$ b) $\left(\sqrt[3]{-1.5}\right)^2$ c) $\sqrt[4]{\left(\frac{9}{5}\right)^5}$
    d) $\sqrt[3]{\left(\frac{3}{8}\right)^4}$ e) $\left(\sqrt{\frac{5}{4}}\right)^3$ f) $\sqrt[5]{(-2.5)^3}$

12. Evaluate each power without using a calculator.
    a) $9^{\frac{3}{2}}$ b) $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ c) $(-27)^{\frac{2}{3}}$
    d) $0.36^{1.5}$ e) $(-64)^{\frac{2}{3}}$ f) $\left(\frac{4}{25}\right)^{\frac{3}{2}}$

13. Write an equivalent form for each number using a power with exponent $\frac{1}{2}$, then write the answer as a radical.
    a) 2 b) 4 c) 10 d) 3 e) 5

14. Write an equivalent form for each number using a power with exponent $\frac{1}{3}$, then write the answer as a radical.
    a) -1 b) 2 c) 3 d) -4 e) 4

15. Arrange these numbers in order from least to greatest. Describe your strategy.
    $\sqrt[3]{4}, 4^{\frac{3}{2}}, 4^2, \left(\frac{1}{4}\right)^{\frac{3}{2}}$

16. a) Evaluate.
    i) $16^{1.5}$ ii) $81^{0.75}$
    iii) $(-32)^{0.8}$ iv) $35^{0.5}$
    v) $1.21^{1.5}$ vi) $\left(\frac{3}{4}\right)^{0.6}$

    b) Which powers in part a could you have evaluated without a calculator? How can you tell before you evaluate?

17. The height, h metres, of a certain species of fir tree can be estimated from the formula $h = 35d^{\frac{2}{3}}$, where d metres is the diameter at the base. Use the formula to determine the approximate height of a fir tree with base diameter 3.2 m.

18. Here is a student's solution for evaluating a power.

$$1.96^{\frac{3}{2}} = \left(\sqrt[3]{1.96}\right)^2$$
$$= (1.2514\ldots)^2$$
$$= 1.5661\ldots$$

Identify the errors the student made. Write a correct solution.

19. A formula for the approximate surface area, SA square metres, of a person's body is $SA = 0.096m^{0.7}$, where m is the person's mass in kilograms. Calculate the surface area of a child with mass 40 kg.

20. Here is an expression for the percent of caffeine that remains in your body n hours after you drink a caffeine beverage:

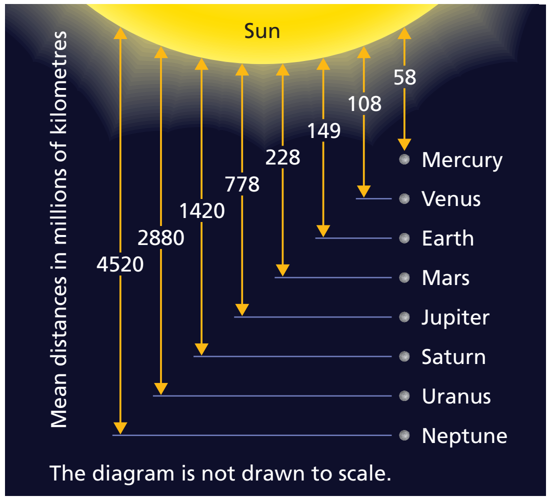
$100(0.5)^{\frac{n}{5}}$

a) Show that this expression and the expression on page 222 give the same result, to the nearest whole number, for the percent of caffeine that remains after $\frac{1}{2}$ h.

b) Use the expression above to determine the percent of caffeine that remains after 1.5 h.

c) After how many hours does 50% of the caffeine remain? Explain how you know.

21. In the late 1500s, Johannes Kepler developed a formula to calculate the time it takes each planet to orbit the sun (called the *period*). The formula is $T \doteq 0.2R^{\frac{3}{2}}$, where T is the period in Earth days and R is the mean distance from the planet to the sun in millions of kilometres.



The mean distance of Earth from the sun is about 149 million kilometres. The mean distance of Mars from the sun is about 228 million kilometres. Which planet has the longer period, Earth or Mars? Justify your answer.

C

22. Two students discussed the meaning of the statement $3.2^{4.2} = 132.3213...$

Luc said: It means 3.2 multiplied by itself 4.2 times is about 132.3213.

Karen said: No, you can't multiply a number 4.2 times. $3.2^{4.2}$ can be written as $3.2^{\frac{42}{10}}$. So the statement means that 42 factors, each equal to the tenth root of 3.2, multiplied together will equal about 132.3213.

Which student is correct? Explain.

Reflect

In the power $x^{\frac{m}{n}}$, m and n are natural numbers and x is a rational number. What does the numerator m represent? What does the denominator n represent? Use an example to explain your answer.

What must be true about x for $x^{\frac{m}{n}}$ to be a rational number?

4.5 Negative Exponents and Reciprocals

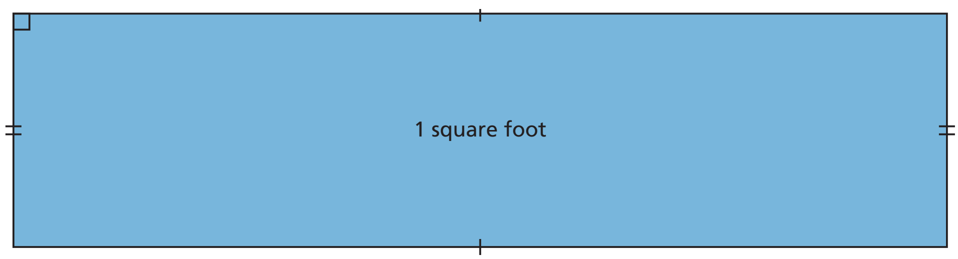
LESSON FOCUS

Relate negative exponents to reciprocals.

Scientists can calculate the speed of a dinosaur from its tracks. These tracks were found near Grand Cache, Alberta.

Make Connections

A rectangle has area 1 square foot.
List 5 possible pairs of lengths and widths for this rectangle.
What is the relationship between the possible lengths and widths?



Construct Understanding

TRY THIS

Work with a partner.

You will need grid paper and scissors.

A. Cut out a 16 by 16 grid. Determine the area of the grid in square units and as a power of 2. Record your results in a table like this:

Cut	Area (units^2)	Area as a Power of 2
Start	256	
1		
2		
3		

B. Cut the grid in half and discard one piece.
In the table, record the area of the remaining piece in square units and as a power of 2.

C. Repeat Step B until the paper cannot be cut further.

D. Use patterns to extend the second and third columns of the table to Cut 13.

E. Compare the areas for each pair of powers in the table:
- 2^{-1} and 2^1
- 2^{-2} and 2^2
- 2^{-3} and 2^3

What relationships do you notice?

Two numbers with a product of 1 are reciprocals.
Since $4 \cdot \frac{1}{4} = 1$, the numbers 4 and $\frac{1}{4}$ are reciprocals.

Similarly, $\frac{2}{3} \cdot \frac{3}{2} = 1$, so the numbers $\frac{2}{3}$ and $\frac{3}{2}$ are also reciprocals.

We define powers with negative exponents so that previously developed properties such as $a^m \cdot a^n = a^{m+n}$ and $a^0 = 1$ still apply.

Apply these properties.

$$\begin{aligned} 5^{-2} \cdot 5^2 &= 5^{-2+2} \\ &= 5^0 \\ &= 1 \end{aligned}$$

Since the product of 5^{-2} and 5^2 is 1, 5^{-2} and 5^2 are reciprocals.

So, $5^{-2} = \frac{1}{5^2}$ and $\frac{1}{5^{-2}} = 5^2$

That is, $5^{-2} = \frac{1}{25}$

This suggests the following definition for powers with negative exponents.

Powers with Negative Exponents

When x is any non-zero number and n is a rational number, x^{-n} is the reciprocal of x^n.

That is, $x^{-n} = \frac{1}{x^n}$ and $\frac{1}{x^{-n}} = x^n,\ x \neq 0$

Why can't x be 0?

Example 1 Evaluating Powers with Negative Integer Exponents

Evaluate each power.

a) 3^{-2} **b)** $\left(-\frac{3}{4}\right)^{-3}$ **c)** 0.3^{-4}

SOLUTION

a) $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$

b) $\left(-\frac{3}{4}\right)^{-3} = \left(-\frac{4}{3}\right)^3$

$= -\frac{64}{27}$

c) 0.3^{-4}

Use a calculator.

```
0.3^-4
123.4567901
```

$0.3^{-4} = 123.4567\ldots$

CHECK YOUR UNDERSTANDING

1. Evaluate each power.

a) 7^{-2} **b)** $\left(\frac{10}{3}\right)^{-3}$

c) $(-1.5)^{-3}$

[Answers: a) $\frac{1}{49}$ b) $\frac{27}{1000}$ c) $-0.2962\ldots$]

We can apply the meaning of rational exponents and negative exponents to evaluate powers with negative rational exponents. For example, the rational exponent in the power $8^{-\frac{2}{3}}$ indicates these operations at the right.

reciprocal ↓ $8^{-\frac{2}{3}}$ ← square (2) ← cube root (3)

Since the exponent $-\frac{2}{3}$ is the product: $(-1)\left(\frac{1}{3}\right)(2)$, and order does not matter when we multiply, we can apply the three operations of reciprocal, cube root, and square in any order.

Example 2 Evaluating Powers with Negative Rational Exponents

Evaluate each power without using a calculator.

a) $8^{-\frac{2}{3}}$ **b)** $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

(Solution continues.)

SOLUTION

a) $8^{-\frac{2}{3}} = \dfrac{1}{8^{\frac{2}{3}}}$ Write with a positive exponent.

$= \dfrac{1}{(\sqrt[3]{8})^2}$ Take the cube root.

$= \dfrac{1}{2^2}$ Square the result.

$= \dfrac{1}{4}$

b) $\left(\dfrac{9}{16}\right)^{-\frac{3}{2}} = \left(\dfrac{16}{9}\right)^{\frac{3}{2}}$ Write with a positive exponent.

$= \left(\sqrt{\dfrac{16}{9}}\right)^3$ Take the square root.

$= \left(\dfrac{4}{3}\right)^3$ Cube the result.

$= \dfrac{64}{27}$

CHECK YOUR UNDERSTANDING

2. Evaluate each power without using a calculator.

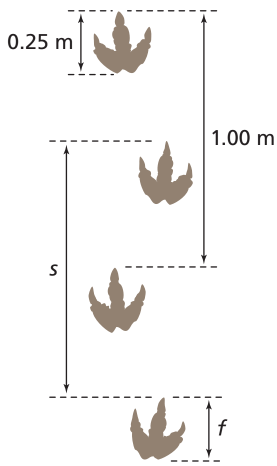
a) $16^{-\frac{5}{4}}$ **b)** $\left(\dfrac{25}{36}\right)^{-\frac{1}{2}}$

[Answers: a) $\dfrac{1}{32}$ b) $\dfrac{6}{5}$]

What other strategies could you use to evaluate the powers?

Example 3 Applying Negative Exponents

Paleontologists use measurements from fossilized dinosaur tracks and the formula $v = 0.155\,s^{\frac{5}{3}}f^{-\frac{7}{6}}$ to estimate the speed at which the dinosaur travelled. In the formula, v is the speed in metres per second, s is the distance between successive footprints of the same foot, and f is the foot length in metres.
Use the measurements in the diagram to estimate the speed of the dinosaur.



SOLUTION

Use the formula: $v = 0.155\,s^{\frac{5}{3}}f^{-\frac{7}{6}}$

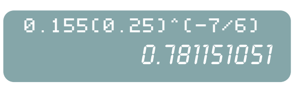
Substitute: $s = 1$ and $f = 0.25$

$v = 0.155\,(1)^{\frac{5}{3}}\,(0.25)^{-\frac{7}{6}}$

$v = 0.155\,(0.25)^{-\frac{7}{6}}$

$v = 0.7811\ldots$

The dinosaur travelled at approximately 0.8 m/s.



CHECK YOUR UNDERSTANDING

3. Use the formula $v = 0.155\,s^{\frac{5}{3}}f^{-\frac{7}{6}}$ to estimate the speed of a dinosaur when $s = 1.5$ and $f = 0.3$.

[Answer: approximately 1.2 m/s]

What is the speed of the dinosaur in kilometres per hour?

Discuss the Ideas

1. When m is an integer, describe the relationship between a^m and a^{-m}.

2. Why is there usually more than one way to determine the value of a power of the form $a^{-\frac{m}{n}}$? Use examples to justify your answer.

Exercises

A

3. Copy then complete each equation.

   a) $\frac{1}{5^4} = 5^{\square}$ b) $\left(-\frac{1}{2}\right)^{-3} = (-2)^{\square}$

   c) $\frac{1}{3^{\square}} = 3^2$ d) $\frac{1}{4^{-2}} = 4^{\square}$

4. Evaluate the powers in each pair without a calculator.

   a) 4^2 and 4^{-2} b) 2^4 and 2^{-4}

   c) 6^1 and 6^{-1} d) 4^3 and 4^{-3}

   Describe what is similar about the answers, and what is different.

5. Given that $2^{10} = 1024$, what is 2^{-10}?

6. Write each power with a positive exponent.

   a) 2^{-3} b) 3^{-5} c) $(-7)^{-2}$

7. Write each power with a positive exponent.

   a) $\left(\frac{1}{2}\right)^{-2}$ b) $\left(\frac{2}{3}\right)^{-3}$ c) $\left(-\frac{6}{5}\right)^{-4}$

8. Evaluate each power without using a calculator.

   a) 3^{-2} b) 2^{-4} c) $(-2)^{-5}$

   d) $\left(\frac{1}{3}\right)^{-3}$ e) $\left(-\frac{2}{3}\right)^{-2}$ f) $\frac{1}{5^{-3}}$

B

9. Evaluate each power without using a calculator.

   a) $4^{-\frac{1}{2}}$ b) $0.09^{-\frac{1}{2}}$

   c) $27^{-\frac{1}{3}}$ d) $(-64)^{-\frac{1}{3}}$

   e) $(-0.027)^{-\frac{2}{3}}$ f) $32^{-\frac{2}{5}}$

   g) $9^{-\frac{3}{2}}$ h) $0.04^{-\frac{3}{2}}$

10. Use a power with a negative exponent to write an equivalent form for each number.

    a) $\frac{1}{9}$ b) $\frac{1}{5}$ c) 4 d) -3

11. When you save money in a bank, the bank pays you *interest*. This interest is added to your investment and the resulting amount also earns interest. We say the interest *compounds*. Suppose you want an amount of \$3000 in 5 years. The interest rate for the savings account is 2.5% compounded annually. The money, P dollars, you must invest now is given by the formula: $P = 3000(1.025)^{-5}$. How much must you invest now to have \$3000 in 5 years?

12. Here is a student's solution for evaluating a power. Identify any errors in the solution. Write a correct solution.

$$\left(-\frac{64}{125}\right)^{-\frac{5}{3}} = \left(\frac{64}{125}\right)^{\frac{5}{3}}$$
$$= \left(\sqrt[3]{\frac{64}{125}}\right)^5$$
$$= \left(\frac{4}{5}\right)^5$$
$$= \frac{1024}{3125}$$

13. Evaluate each power without using a calculator.

    a) $27^{-\frac{4}{3}}$ b) $16^{-1.5}$ c) $32^{-0.4}$

    d) $\left(-\frac{8}{27}\right)^{-\frac{2}{3}}$ e) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$ f) $\left(\frac{9}{4}\right)^{-\frac{5}{2}}$

14. Michelle wants to invest enough money on January 1st to pay her nephew \$150 at the end of each year for the next 10 years. The savings account pays 3.2% compounded annually. The money, P dollars, that Michelle must invest today is given by the formula $P = \frac{150\left[1 - 1.032^{-10}\right]}{0.032}$. How much must Michelle invest on January 1st?

15. The intensity of light at its source is 100%. The intensity, I, at a distance d centimetres from the source is given by the formula $I = 100d^{-2}$. Use the formula to determine the intensity of the light 23 cm from the source.

16. Which is greater, 2^{-5} or 5^{-2}? Verify your answer.

17. a) Identify the patterns in this list.

$16 = 2^4$
$8 = 2^3$
$4 = 2^2$

b) Extend the patterns in part a downward. Write the next 5 rows in the pattern.

c) Explain how this pattern shows that $a^{-n} = \frac{1}{a^n}$.

18. How many times as great as 3^{-5} is 3^3? Express your answer as a power and in standard form.

19. What do you know about the sign of the exponent in each case? Justify your answers.

a) $3^x > 1$ b) $3^x < 1$ c) $3^x = 1$

C

20. A number is raised to a negative exponent. Is it always true that the value of the power will be less than 1? Use an example to explain.

21. There is a gravitational force, F newtons, between Earth and the moon. This force is given by the formula $F = (6.67 \times 10^{-11})Mmr^{-2}$, where M is the mass of Earth in kilograms, m is the mass of the moon in kilograms, and r is the distance between Earth and the moon in metres. The mass of Earth is approximately 5.9736×10^{24} kg. The mass of the moon is approximately 7.349×10^{22} kg. The mean distance between them is approximately 382 260 km.

a) What is the gravitational force between Earth and the moon?

b) The value r is actually the distance between the centres of Earth and the moon. Research to find the diameters of Earth and the moon. Calculate the gravitational force with this new value of r.

Reflect

Explain what a negative exponent means. Use examples to demonstrate your thinking.

THE WORLD OF MATH

Math Fact: Computers and π

The most famous irrational number of all, π, was not proved irrational until the 18th century. Throughout history, many mathematicians have spent much time manually calculating the decimal digits of π. This had no practical value, but did provide motivation for some discoveries that have been influential in modern mathematics. In 1949, an early computer calculated π to 2000 digits, which was many more digits than any previous manual calculations. Today, calculating digits of π is a test of the power, speed, and accuracy of supercomputers. Computers have calculated π to more than 6 billion digits, and even a home computer can be programmed to calculate π to millions of digits in a few hours. This T-shirt shows the first 100 digits of π.

CHECKPOINT 2

Connections

$$(a^m)^n = a^{mn}$$

When $m = \frac{1}{n}$

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \cdot n} = a^1 = a$$

Definition of a radical

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$ and $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}}$

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

$$a^n \cdot a^m = a^{n+m}$$

When $m = -n$

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$$

Definition of a reciprocal

$$a^{-n} = \frac{1}{a^n}$$

When n is a rational number

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$$

Reciprocal of a fraction

$$\left(\frac{a}{b}\right)^{-\frac{m}{n}} = \left(\frac{b}{a}\right)^{\frac{m}{n}}$$

Concept Development

- **In Lesson 4.4**
  - You used patterns to explain a meaning for an exponent of the form $\frac{1}{n}$.
  - You applied the exponent law for the power of a power to justify why a **power with exponent $\frac{1}{n}$ is the *n*th root of the base of the power.**
- **In Lesson 4.5**
  - You used patterns to explain a meaning for a negative exponent.
  - You applied the exponent law for multiplying powers to justify why a **power with a negative rational exponent is written as a reciprocal.**

Assess Your Understanding

4.4

1. Evaluate each power without using a calculator.

a) $16^{\frac{1}{4}}$ **b)** $49^{0.5}$ **c)** $(-64)^{\frac{2}{3}}$ **d)** $\left(\frac{49}{9}\right)^{1.5}$ **e)** $(-8)^{\frac{5}{3}}$

2. a) Write each power as a radical.

i) $35^{\frac{2}{3}}$ **ii)** $32^{\frac{3}{2}}$ **iii)** $(-32)^{\frac{2}{5}}$

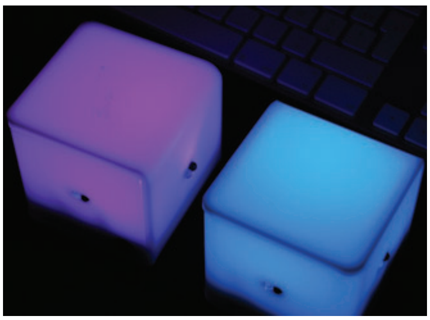
iv) $400^{1.5}$ **v)** $(-125)^{\frac{1}{3}}$ **vi)** $\left(\frac{8}{125}\right)^{\frac{2}{3}}$

b) Evaluate each radical in part a without using a calculator, if possible. Explain why you could not evaluate some radicals.

3. Write each radical in exponent form.

a) $\sqrt[3]{4}$ **b)** $\sqrt{9}$ **c)** $\sqrt[4]{18}$ **d)** $\left(\sqrt{10}\right)^3$ **e)** $\left(\sqrt[3]{-10}\right)^2$

4. The circulation time is the average time it takes for all the blood in the body to circulate once and return to the heart. The circulation time for a mammal can be estimated from the formula $T \doteq 17.4m^{\frac{1}{4}}$, where T is the circulation time in seconds and m is the body mass in kilograms. Estimate the circulation time for a mammal with mass 85 kg.



5. Arrange these numbers in order from least to greatest.

$3^{\frac{3}{2}}, \sqrt[3]{3}, \left(\sqrt{3}\right)^5, 3^{\frac{2}{3}}, \left(\sqrt[3]{3}\right)^4$

6. An AudioCube creates sounds and musical patterns. Each cube exchanges musical information wirelessly with nearby cubes. The volume of an AudioCube is 421 875 mm^3. Write the edge length of the cube as a radical and as a power, then calculate the edge length.

4.5

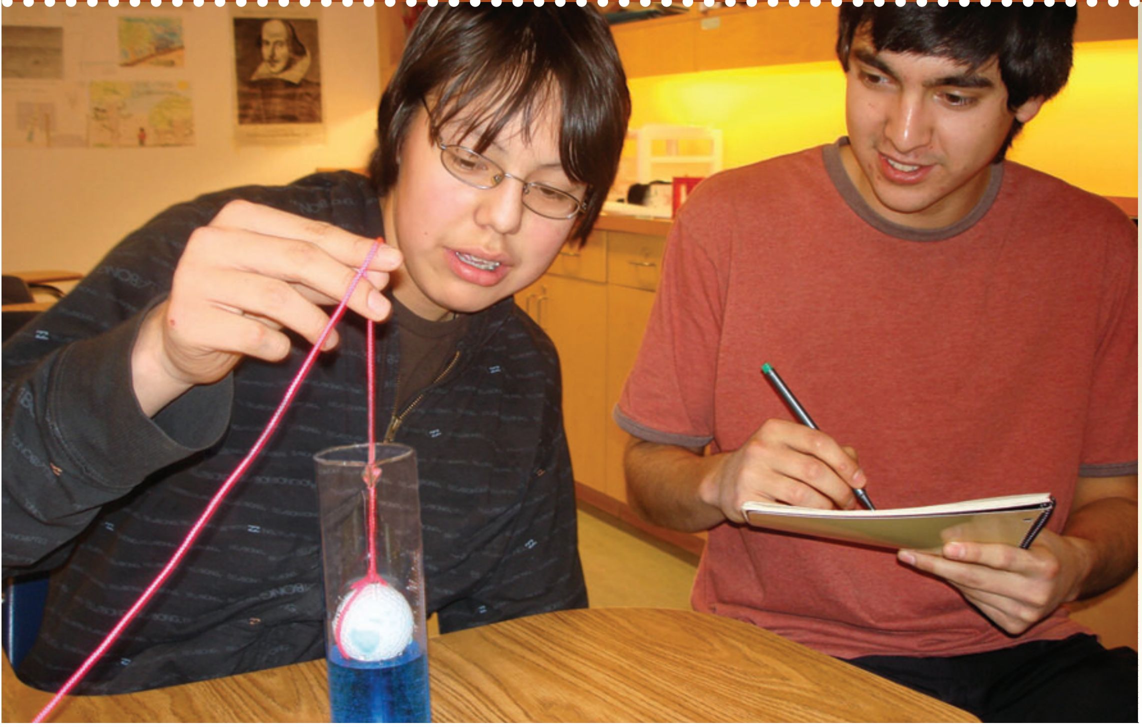
7. Evaluate each power without using a calculator.

a) $\left(\frac{2}{3}\right)^{-4}$ **b)** 0.5^{-2} **c)** $(-1000)^{-\frac{2}{3}}$

d) $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$ **e)** $\left(\frac{1}{10}\right)^{-2}$ **f)** $(-0.008)^{-\frac{4}{3}}$

8. Suppose you want \$5000 in 3 years. The interest rate for a savings account is 2.9% compounded annually. The money, P dollars, you must invest now is given by the formula: $P = 5000(1.029)^{-3}$
How much must you invest now to have \$5000 in 3 years?

4.6 Applying the Exponent Laws



LESSON FOCUS

Apply the exponent laws to simplify expressions.

We can use a measuring cylinder to determine the volume of a sphere. Then we can use the exponent laws to help calculate the radius.

Make Connections

Recall the exponent laws for integer bases and whole number exponents.

Product of powers: $a^m \cdot a^n = a^{m+n}$

Quotient of powers: $a^m \div a^n = a^{m-n}$, $a \neq 0$

Power of a power: $(a^m)^n = a^{mn}$

Power of a product: $(ab)^m = a^m b^m$

Power of a quotient: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$

What other types of numbers could be a base? An exponent?
How would you use the exponent laws to evaluate an expression with these numbers?

Construct Understanding

THINK ABOUT IT

Work on your own.

What is the value of $\left(\frac{a^6b^9}{a^5b^8}\right)^{-2}$ when $a = -3$ and $b = 2$?

Compare strategies with a classmate.
If you used the same strategy, find a different strategy.
Which strategy is more efficient, and why?

We can use the exponent laws to simplify expressions that contain rational number bases. It is a convention to write a simplified power with a positive exponent.

Example 1 Simplifying Numerical Expressions with Rational Number Bases

Simplify by writing as a single power. Explain the reasoning.

a) $0.3^{-3} \cdot 0.3^5$

b) $\left[\left(-\frac{3}{2}\right)^{-4}\right]^2 \cdot \left[\left(-\frac{3}{2}\right)^2\right]^3$

c) $\frac{(1.4^3)(1.4^4)}{1.4^{-2}}$

d) $\left(\frac{7^{\frac{2}{3}}}{7^{\frac{1}{3}} \cdot 7^{\frac{5}{3}}}\right)^6$

SOLUTION

a) $0.3^{-3} \cdot 0.3^5$

Use the product of powers law:

When the bases are the same, add the exponents.

$$0.3^{-3} \cdot 0.3^5 = 0.3^{(-3)+5}$$
$$= 0.3^2$$

b) $\left[\left(-\frac{3}{2}\right)^{-4}\right]^2 \cdot \left[\left(-\frac{3}{2}\right)^2\right]^3$

First use the power of a power law:

For each power, multiply the exponents.

$$\left[\left(-\frac{3}{2}\right)^{-4}\right]^2 \cdot \left[\left(-\frac{3}{2}\right)^2\right]^3 = \left(-\frac{3}{2}\right)^{(-4)(2)} \cdot \left(-\frac{3}{2}\right)^{(2)(3)}$$

Then use the product of powers law.

$$\left[\left(-\frac{3}{2}\right)^{-4}\right]^2 \cdot \left[\left(-\frac{3}{2}\right)^2\right]^3 = \left(-\frac{3}{2}\right)^{-8} \cdot \left(-\frac{3}{2}\right)^6$$

$$= \left(-\frac{3}{2}\right)^{-2}$$ Write with a positive exponent.

$$= \left(-\frac{2}{3}\right)^2$$

c) $\frac{(1.4^3)(1.4^4)}{1.4^{-2}}$ Use the product of powers law.

$$= \frac{1.4^{3+4}}{1.4^{-2}}$$

$$= \frac{1.4^7}{1.4^{-2}}$$ Use the quotient of powers law.

$$= 1.4^{7-(-2)}$$
$$= 1.4^9$$

(Solution continues.)

CHECK YOUR UNDERSTANDING

1. Simplify by writing as a single power. Explain your reasoning.

a) $0.8^2 \cdot 0.8^{-7}$

b) $\left[\left(-\frac{4}{5}\right)^2\right]^{-3} \div \left[\left(-\frac{4}{5}\right)^4\right]^{-5}$

c) $\frac{(1.5^{-3})^{-5}}{1.5^5}$

d) $\frac{9^{\frac{5}{4}} \cdot 9^{-\frac{1}{4}}}{9^{\frac{3}{4}}}$

[Answers: a) $\frac{1}{0.8^5}$ b) $\left(-\frac{4}{5}\right)^{14}$ c) 1.5^{10} d) $9^{\frac{1}{4}}$]

d) $\left(\dfrac{7^{\frac{2}{3}}}{7^{\frac{1}{3}} \cdot 7^{\frac{5}{3}}}\right)^6$ Use the product of powers law.

$= \left(\dfrac{7^{\frac{2}{3}}}{7^{\frac{1}{3}+\frac{5}{3}}}\right)^6$

$= \left(\dfrac{7^{\frac{2}{3}}}{7^{\frac{6}{3}}}\right)^6$ Use the quotient of powers law.

$= \left(7^{\frac{2}{3}-\frac{6}{3}}\right)^6$

$= \left(7^{-\frac{4}{3}}\right)^6$ Use the power of a power law.

$= 7^{\left(-\frac{4}{3}\right)(6)}$

$= 7^{-\frac{24}{3}}$

$= 7^{-8}$ Write with a positive exponent.

$= \dfrac{1}{7^8}$

In part d, what other strategy could you use? Which strategy is more efficient?

Example 2 Simplifying Algebraic Expressions with Integer Exponents

Simplify. Explain the reasoning.

a) $(x^3y^2)(x^2y^{-4})$ **b)** $\dfrac{10a^5b^3}{2a^2b^{-2}}$

SOLUTION

a) $(x^3y^2)(x^2y^{-4}) = x^3 \cdot y^2 \cdot x^2 \cdot y^{-4}$ x^3y^2 means $x^3 \cdot y^2$

$= x^3 \cdot x^2 \cdot y^2 \cdot y^{-4}$ Use the product of powers law.

$= x^{3+2} \cdot y^{2+(-4)}$

$= x^5 \cdot y^{-2}$ Write with a positive exponent.

$= x^5 \cdot \dfrac{1}{y^2}$

$= \dfrac{x^5}{y^2}$

b) $\dfrac{10a^5b^3}{2a^2b^{-2}} = \dfrac{10}{2} \cdot \dfrac{a^5}{a^2} \cdot \dfrac{b^3}{b^{-2}}$ Use the quotient of powers law.

$= 5 \cdot a^{5-2} \cdot b^{3-(-2)}$

$= 5 \cdot a^3 \cdot b^5$

$= 5a^3b^5$

CHECK YOUR UNDERSTANDING

2. Simplify. Explain your reasoning.

a) $m^4n^{-2} \cdot m^2n^3$

b) $\dfrac{6x^4y^{-3}}{14xy^2}$

[Answers: a) m^6n b) $\dfrac{3x^3}{7y^5}$]

Example 3 Simplifying Algebraic Expressions with Rational Exponents

Simplify. Explain the reasoning.

a) $(8a^3b^6)^{\frac{1}{3}}$ b) $(x^{\frac{3}{2}}y^2)(x^{\frac{1}{2}}y^{-1})$

c) $\dfrac{4a^{-2}b^{\frac{2}{3}}}{2a^2b^{\frac{1}{3}}}$ d) $\left(\dfrac{100a}{25a^5b^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$

SOLUTION

a) $(8a^3b^6)^{\frac{1}{3}} = 8^{\frac{1}{3}} \cdot a^{3\left(\frac{1}{3}\right)} \cdot b^{6\left(\frac{1}{3}\right)}$ Using the power of a power law.

$= (2^3)^{\frac{1}{3}} \cdot a^1 \cdot b^2$

$= 2ab^2$

b) $(x^{\frac{3}{2}}y^2)(x^{\frac{1}{2}}y^{-1}) = x^{\frac{3}{2}} \cdot x^{\frac{1}{2}} \cdot y^2 \cdot y^{-1}$ Use the product of powers law.

$= x^{\frac{3}{2}+\frac{1}{2}} \cdot y^{2+(-1)}$

$= x^2y$

c) $\dfrac{4a^{-2}b^{\frac{2}{3}}}{2a^2b^{\frac{1}{3}}} = \dfrac{4}{2} \cdot \dfrac{a^{-2}}{a^2} \cdot \dfrac{b^{\frac{2}{3}}}{b^{\frac{1}{3}}}$ Use the quotient of powers law.

$= 2 \cdot a^{(-2)-2} \cdot b^{\frac{2}{3}-\frac{1}{3}}$

$= 2 \cdot a^{-4} \cdot b^{\frac{1}{3}}$ Write with a positive exponent.

$= \dfrac{2b^{\frac{1}{3}}}{a^4}$

d) $\left(\dfrac{100a}{25a^5b^{-\frac{1}{2}}}\right)^{\frac{1}{2}} = \left(\dfrac{100}{25} \cdot \dfrac{a^1}{a^5} \cdot \dfrac{1}{b^{-\frac{1}{2}}}\right)^{\frac{1}{2}}$ Simplify inside the brackets first. Use the quotient of powers law. Write with a positive exponent.

$= \left(4 \cdot a^{1-5} \cdot b^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$= \left(4 \cdot a^{-4} \cdot b^{\frac{1}{2}}\right)^{\frac{1}{2}}$ Use the power of a power law.

$= 4^{\frac{1}{2}} \cdot a^{(-4)\left(\frac{1}{2}\right)} \cdot b^{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}$

$= 2 \cdot a^{-2} \cdot b^{\frac{1}{4}}$ Write with a positive exponent.

$= \dfrac{2b^{\frac{1}{4}}}{a^2}$

CHECK YOUR UNDERSTANDING

3. Simplify. Explain your reasoning.

a) $(25a^4b^2)^{\frac{3}{2}}$

b) $(x^3y^{-\frac{3}{2}})(x^{-1}y^{\frac{1}{2}})$

c) $\dfrac{12x^{-5}y^{\frac{5}{2}}}{3x^{\frac{1}{2}}y^{-\frac{1}{2}}}$

d) $\left(\dfrac{50x^2y^4}{2x^4y^7}\right)^{\frac{1}{2}}$

[Answers: a) $125a^6b^3$ b) $\dfrac{x^2}{y}$ c) $\dfrac{4y^3}{x^{\frac{11}{2}}}$ d) $\dfrac{5}{xy^{\frac{3}{2}}}$]

Example 4 Solving Problems Using the Exponent Laws

A sphere has volume 425 m^3.
What is the radius of the sphere to the nearest tenth of a metre?

SOLUTION

The volume V of a sphere with radius r is given by the formula: $V = \frac{4}{3}\pi r^3$. Substitute $V = 425$, then solve for r.

$$425 = \frac{4}{3}\pi r^3$$ Multiply each side by 3.

$$3(425) = 3\left(\frac{4}{3}\pi r^3\right)$$

$$1275 = 4\pi r^3$$ Divide each side by 4π.

$$\frac{1275}{4\pi} = \frac{4\pi r^3}{4\pi}$$

$$\frac{1275}{4\pi} = r^3$$ To solve for r, take the cube root of each side by raising each side to the one-third power.

$$\left(\frac{1275}{4\pi}\right)^{\frac{1}{3}} = (r^3)^{\frac{1}{3}}$$ Use the power of a power law.

$$\left(\frac{1275}{4\pi}\right)^{\frac{1}{3}} = r$$

```
3√(1275/(4π))
      4.664088405
```

$$r = 4.6640\ldots$$

The radius of the sphere is approximately 4.7 m.

CHECK YOUR UNDERSTANDING

4. A cone with height and radius equal has volume 18 cm^3. What are the radius and height of the cone to the nearest tenth of a centimetre?

[Answer: approximately 2.6 cm]

How do you know that the length of the radius is an irrational number?

Discuss the Ideas

1. Suppose you want to evaluate an algebraic expression for particular values of the variables. Why might it be helpful to simplify the expression first?

2. When you simplify an expression, how do you know which exponent law to apply first?

Exercises

Write all powers with positive exponents.

A

3. Simplify.

a) $x^3 \cdot x^4$ b) $a^2 \cdot a^{-5}$

c) $b^{-3} \cdot b^5$ d) $m^2 \cdot m^{-3}$

4. Write as a single power.

a) $0.5^2 \cdot 0.5^3$ b) $0.5^2 \cdot 0.5^{-3}$

c) $\frac{0.5^2}{0.5^3}$ d) $\frac{0.5^2}{0.5^{-3}}$

5. Simplify.

a) $\frac{x^4}{x^2}$ b) $\frac{x^2}{x^5}$

c) $n^6 \div n^5$ d) $\frac{a^2}{a^6}$

6. Simplify.

a) $(n^2)^3$ b) $(z^2)^{-3}$

c) $(n^{-4})^{-3}$ d) $(c^{-2})^2$

7. Write as a single power.

a) $\left[\left(\frac{3}{5}\right)^3\right]^4$ b) $\left[\left(\frac{3}{5}\right)^3\right]^{-4}$

c) $\left[\left(\frac{3}{5}\right)^{-3}\right]^{-4}$ d) $\left[\left(-\frac{3}{5}\right)^{-3}\right]^{-4}$

8. Simplify.

a) $\left(\frac{a}{b}\right)^2$ b) $\left(\frac{n^2}{m}\right)^3$

c) $\left(\frac{c^2}{d^2}\right)^{-4}$ d) $\left(\frac{2b}{5c}\right)^2$

e) $(ab)^2$ f) $(n^2m)^3$

g) $(c^3d^2)^{-4}$ h) $(xy^{-1})^3$

B

9. Simplify. State the exponent law you used.

a) $x^{-3} \cdot x^4$ b) $a^{-4} \cdot a^{-1}$

c) $b^4 \cdot b^{-3} \cdot b^2$ d) $m^8 \cdot m^{-2} \cdot m^{-6}$

e) $\frac{x^{-5}}{x^2}$ f) $\frac{s^5}{s^{-5}}$

g) $\frac{b^{-8}}{b^{-3}}$ h) $\frac{t^{-4}}{t^{-4}}$

10. Evaluate.

a) $1.5^{\frac{3}{2}} \cdot 1.5^{\frac{1}{2}}$ b) $\left(\frac{3}{4}\right)^{\frac{3}{4}} \cdot \left(\frac{3}{4}\right)^{\frac{5}{4}}$

c) $(-0.6)^{\frac{1}{3}} \cdot (-0.6)^{\frac{5}{3}}$ d) $\left(\frac{4}{5}\right)^{\frac{4}{3}} \cdot \left(\frac{4}{5}\right)^{-\frac{4}{3}}$

e) $\frac{0.6^{\frac{1}{2}}}{0.6^{\frac{3}{2}}}$ f) $\frac{\left(-\frac{3}{8}\right)^{\frac{2}{3}}}{\left(-\frac{3}{8}\right)^{-\frac{1}{3}}}$

g) $\frac{0.49^{\frac{5}{2}}}{0.49^{4}}$ h) $\frac{0.027^{\frac{5}{3}}}{0.027^{\frac{4}{3}}}$

11. Simplify. Explain your reasoning.

a) $(x^{-1}y^{-2})^{-3}$ b) $(2a^{-2}b^2)^{-2}$

c) $(4m^2n^3)^{-3}$ d) $\left(\frac{3}{2}m^{-2}n^{-3}\right)^{-4}$

12. A cone with equal height and radius has volume 1234 cm^3. What is the height of the cone to the nearest tenth of a centimetre?

13. A sphere has volume 375 cubic feet. What is the surface area of the sphere to the nearest square foot?

14. Simplify. Which exponent laws did you use?

a) $\frac{(a^2b^{-1})^{-2}}{(a^{-3}b)^3}$ b) $\left(\frac{(c^{-3}d)^{-1}}{c^2d}\right)^{-2}$

15. Evaluate each expression for $a = -2$ and $b = 1$. Explain your strategy.

a) $(a^3b^2)(a^2b^3)$ b) $(a^{-1}b^{-2})(a^{-2}b^{-3})$

c) $\frac{a^{-4}b^5}{ab^3}$ d) $\left(\frac{a^{-7}b^7}{a^{-9}b^{10}}\right)^{-5}$

16. Simplify.

a) $m^{\frac{2}{3}} \cdot m^{\frac{4}{3}}$ b) $x^{-\frac{3}{2}} \div x^{-\frac{1}{4}}$

c) $\frac{-9a^{-4}b^{\frac{3}{4}}}{3a^2b^{\frac{1}{4}}}$ d) $\left(\frac{-64c^6}{a^9b^{-\frac{1}{2}}}\right)^{\frac{1}{3}}$

17. Identify any errors in each solution for simplifying an expression. Write a correct solution.

a) $(x^2y^{-3})(x^{\frac{1}{2}}y^{-1}) = x^2 \cdot x^{\frac{1}{2}} \cdot y^{-3} \cdot y^{-1}$
$= x^1 \cdot y^3$
$= xy^3$

b) $\left(\frac{-5a^2}{b^{\frac{1}{2}}}\right)^{-2} = \frac{10a^{-4}}{b^{-1}}$
$= \frac{10b}{a^4}$

18. Explain how to use a measuring cylinder containing water to calculate the diameter of a marble that fits inside the cylinder.

19. Identify the errors in each simplification. Write the correct solution.

a) $\frac{(m^{-3} . n^2)^{-4}}{(m^2 . n^{-3})^2} = (m^{-5} . n^5)^{-6}$
$= m^{30} . n^{30}$
$= (mn)^{30}$

b) $\left(r^{\frac{1}{2}} . s^{-\frac{3}{2}}\right)^{\frac{1}{2}} . \left(r^{-\frac{1}{4}} . s^{\frac{1}{2}}\right)^{-1} = r^1 . s^{-1} . r^{-\frac{5}{4}} . s^{-\frac{1}{2}}$
$= r^{1-\frac{5}{4}} . s^{-1-\frac{1}{2}}$
$= r^{-\frac{1}{4}} . s^{-\frac{3}{2}}$
$= \frac{1}{r^{\frac{1}{4}} . s^{\frac{3}{2}}}$

20. ISO paper sizes A0, A1, A2, …, are commonly used outside of North America. For any whole number n, the width, in metres, of a piece of An paper is $2^{-\frac{2n+1}{4}}$ and its length, in metres, is $2^{-\frac{2n-1}{4}}$.

a) Write, then simplify expressions to represent the dimensions of each piece of paper. Evaluate each measure to the nearest millimetre.
   i) A3   ii) A4   iii) A5

b) Suppose each piece of paper from part a is folded in half along a line perpendicular to its length. Write, then simplify expressions to represent the dimensions of each folded piece.

c) Compare your results in parts a and b. What do you notice?

C

21. Simplify. Show your work.

a) $\left(\dfrac{a^{-3}b}{c^2}\right)^{-4} \cdot \left(\dfrac{c^5}{a^4b^{-3}}\right)^{-1}$ b) $\dfrac{(2a^{-1}b^4c^{-3})^{-2}}{(4a^2bc^{-4})^2}$

22. If $x = a^{-2}$ and $y = a^{\frac{2}{3}}$, write each expression in terms of a.

a) $\left(x^{\frac{1}{2}}y^{\frac{2}{3}}\right)^2$ b) $\left(x^{\frac{3}{4}} \div y^{-\frac{1}{2}}\right)^3$

23. Write 3 different expressions for each result.

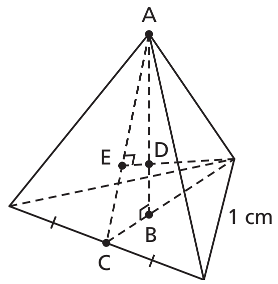
a) $x^{\frac{3}{2}}$ is the product of two powers with rational exponents.

b) $x^{\frac{3}{2}}$ is the quotient of two powers with rational exponents.

c) $x^{\frac{3}{2}}$ is the result of raising a power with a rational exponent to a rational exponent.

24. A regular tetrahedron has edge length 1 cm. It is placed inside a sphere so that all its vertices touch the surface of the sphere. Point D is the centre of the sphere. The measures, in centimetres, of 3 line segments are:

$$AB = \left(\frac{2}{3}\right)^{\frac{1}{2}};\ AC = \frac{3}{2}\left(\frac{1}{3}\right)^{\frac{1}{2}};\ AE = \left(\frac{1}{3}\right)^{\frac{1}{2}}$$



Given that △ABC is similar to △AED and $\dfrac{AC}{AB} = \dfrac{AD}{AE}$; determine the length of AD.

Reflect

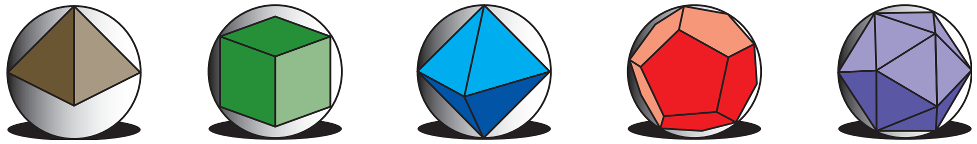
Explain how to apply the exponent laws to simplify algebraic expressions. Use examples to illustrate the types of expressions you can simplify.

THE WORLD OF MATH

Math Fact: Platonic Solids

Platonic solids are the only regular polyhedra that can be placed in a sphere so that each vertex touches the surface of the sphere. In about 300 B.C.E, Euclid used trigonometry, similar triangles, and the Pythagorean Theorem to show the ratio of the edge length of each Platonic solid to the diameter of the sphere:

Tetrahedron   Cube   Octahedron   Dodecahedron   Icosahedron



Tetrahedron: $1 : \left(\frac{3}{2}\right)^{\frac{1}{2}}$

Cube: $1 : 3^{\frac{1}{2}}$

Octahedron: $1 : 2^{\frac{1}{2}}$

Dodecahedron: $1 : \frac{1}{2}\left(3^{\frac{1}{2}} + 15^{\frac{1}{2}}\right)$

Icosahedron: $1 : \frac{1}{2}\left(10 + 2(5)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

STUDY GUIDE

CONCEPT SUMMARY

Big Ideas	Applying the Big Ideas
■ Any number that can be written as the fraction $\frac{m}{n}$, $n \neq 0$, where m and n are integers, is rational.	This means that: ■ If a real number can be expressed as a terminating or repeating decimal, it is rational; otherwise, it is irrational.
■ Exponents can be used to represent roots and reciprocals of rational numbers.	■ The numerator of a rational exponent indicates a power, while the denominator indicates a root. A negative exponent indicates a reciprocal.
■ The exponent laws can be extended to include powers with rational and variable bases, and rational exponents.	■ We can use the exponent laws to simplify expressions that involve rational exponents.

Reflect on the Chapter

- How can you predict whether the value of a radical will be a rational number or an irrational number?
- How were the exponent laws used to create definitions for negative exponents and rational exponents?
- What does it mean to simplify an expression involving radicals or exponents?

THE WORLD OF MATH

Careers: Financial Planner

The financial services industry offers many career opportunities – in banking, insurance, investment firms, as well as private practice. A personal financial planner uses math and technology to explore different ways for a person to invest her or his money. The ability to understand, manipulate, and evaluate algebraic formulas that involve rational exponents is an essential skill.

SKILLS SUMMARY

Skill	Description	Example
Classify numbers. [4.1, 4.2]	To determine whether a number is rational or irrational, write the number in decimal form. ■ Repeating and terminating decimals are rational. ■ Non-repeating, non-terminating decimals are irrational.	Rational numbers: $2, 0, -3, 3.75, 0.0\overline{1}, \frac{3}{5}, -\frac{10}{7}$ Irrational numbers: $\sqrt{3}, \pi$
Simplify radicals. [4.3]	To simplify a square root: **1.** Write the radicand as a product of its greatest perfect square factor and another number. **2.** Take the square root of the perfect square factor. A similar procedure applies for cube roots and higher roots.	$\sqrt{200} = \sqrt{100 \cdot 2}$ $= \sqrt{100} \cdot \sqrt{2}$ $= 10\sqrt{2}$ $\sqrt[3]{200} = \sqrt[3]{8 \cdot 25}$ $= \sqrt[3]{8} \cdot \sqrt[3]{25}$ $= 2\sqrt[3]{25}$
Evaluate powers. [4.4, 4.5]	To evaluate powers without using a calculator: **1.** Rewrite a power with a negative exponent as a power with a positive exponent. **2.** Represent powers with fractional exponents as radicals. **3.** Use mental math to evaluate the powers and/or simplify the roots.	$64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}}$ $= \frac{1}{(\sqrt[3]{64})^2}$ $= \frac{1}{4^2}$ $= \frac{1}{16}$
Apply the exponent laws to simplify expressions. [4.6]	To simplify expressions involving powers: **1.** Remove brackets by applying the exponent laws for products of powers, quotients of powers, or powers of powers. **2.** Write the simplest expression using positive exponents.	$\left(\frac{(xy^2)^3}{x^5y}\right)^{-4} = \left(\frac{x^3y^6}{x^5y}\right)^{-4}$ $= \left(\frac{x^5y^1}{x^3y^6}\right)^4$ $= (x^{5-3}y^{1-6})^4$ $= (x^2y^{-5})^4$ $= x^8y^{-20}$ $= \frac{x^8}{y^{20}}$

REVIEW

4.1

1. Evaluate each radical. Why do you not need a calculator?

a) $\sqrt[3]{1000}$ b) $\sqrt{0.81}$

c) $\sqrt[6]{64}$ d) $\sqrt[4]{\frac{81}{625}}$

2. Explain, using examples, the meaning of the index of a radical.

3. Estimate the value of each radical to 1 decimal place. What strategies can you use?

a) $\sqrt{11}$ b) $\sqrt[3]{-12}$ c) $\sqrt[4]{15}$

4. Identify the number in each case.
a) 5 is a square root of the number.
b) 6 is the cube root of the number.
c) 7 is a fourth root of the number.

5. For $\sqrt[3]{35}$, does its decimal form terminate, repeat, or neither? Support your answer with an explanation.

4.2

6. Tell whether each number is rational or irrational. Justify your answers.

a) -2 b) 17 c) $\sqrt{16}$
d) $\sqrt{32}$ e) 0.756 f) $12.\overline{3}$
g) 0 h) $\sqrt[3]{81}$ i) π

7. Determine the approximate side length of a square with area 23 cm^2. How could you check your answer?

8. Look at this calculator screen.

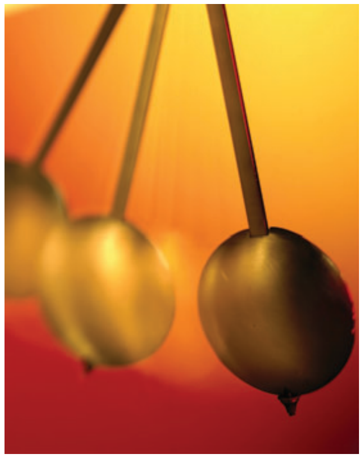
π
3.141592654

a) Is the number 3.141 592 654 rational or irrational? Explain.
b) Is the number π rational or irrational? Explain your answer.

9. Place each number on a number line, then order the numbers from least to greatest.

$\sqrt[3]{30}$, $\sqrt{20}$, $\sqrt[4]{18}$, $\sqrt[3]{-30}$, $\sqrt{30}$, $\sqrt[4]{10}$

10. The formula $T = 2\pi\sqrt{\frac{L}{9.8}}$ gives the time, T seconds, for one complete swing of a pendulum with length L metres. A clock pendulum is 0.25 m long. What time does the pendulum take to complete one swing? Give the answer to the nearest second.



4.3

11. Write each radical in simplest form.

a) $\sqrt{150}$ b) $\sqrt[3]{135}$

c) $\sqrt{112}$ d) $\sqrt[4]{162}$

12. Write each mixed radical as an entire radical.

a) $6\sqrt{5}$ b) $3\sqrt{14}$

c) $4\sqrt[3]{3}$ d) $2\sqrt[4]{2}$

13. Alfalfa cubes are fed to horses to provide protein, minerals, and vitamins.

Two sizes of cubes have volumes 32 cm^3 and 11 cm^3. What is the difference in the edge lengths of the cubes? How can you use radicals to find out?

14. A student simplified $\sqrt{300}$ as shown:

$$\begin{aligned}\sqrt{300} &= \sqrt{3}\cdot\sqrt{100}\\ &= \sqrt{3}\cdot\sqrt{50}\cdot\sqrt{50}\\ &= \sqrt{3}\cdot\sqrt{2}\cdot\sqrt{25}\cdot\sqrt{2}\cdot\sqrt{25}\\ &= 3\cdot 5\cdot\sqrt{2}\cdot 5\\ &= 75\sqrt{2}\end{aligned}$$

Identify the errors the student made, then write a correct solution.

15. Arrange these numbers in order from greatest to least, without using a calculator. Describe your strategy.
$5\sqrt{2}, 4\sqrt{3}, 3\sqrt{6}, 2\sqrt{7}, 6\sqrt{2}$

4.4

16. Show, with examples, why $a^{\frac{1}{n}} = \sqrt[n]{a}$, when n is a natural number and a is a rational number.

17. Express each power as a radical.

a) $12^{\frac{1}{4}}$ b) $(-50)^{\frac{5}{3}}$

c) $1.2^{0.5}$ d) $\left(\frac{3}{8}\right)^{\frac{1}{3}}$

18. Express each radical as a power.

a) $\sqrt{1.4}$ b) $\sqrt[3]{13^2}$

c) $\left(\sqrt[5]{2.5}\right)^4$ d) $\left(\sqrt[4]{\frac{2}{5}}\right)^3$

19. Evaluate each power without using a calculator.

a) $16^{0.25}$ b) $1.44^{\frac{1}{2}}$

c) $(-8)^{\frac{5}{3}}$ d) $\left(\frac{9}{16}\right)^{\frac{3}{2}}$

20. Radioactive isotopes decay. The half-life of an isotope is the time for its mass to decay by $\frac{1}{2}$. For example, polonium-210 has a half-life of 20 weeks. So, a sample of 100 g would decay to 50 g in 20 weeks. The percent, P, of polonium remaining after time t weeks is given by the formula $P = 100(0.5)^{\frac{t}{20}}$. What percent of polonium remains after 30 weeks?

21. Arrange these numbers in order from greatest to least. Describe the strategy you used.
$\sqrt[4]{5}, 5^{\frac{2}{3}}, \sqrt[3]{5}, 5^{\frac{3}{4}}, \left(\sqrt{5}\right)^3$

22. Kleiber's law relates a mammal's metabolic rate while resting, q Calories per day, to its body mass, M kilograms:

$q = 70M^{\frac{3}{4}}$

What is the approximate metabolic rate of each animal?

a) a cow with mass 475 kg

b) a mouse with mass 25 g

4.5

23. a) Identify the patterns in this list.

$81 = 3^4$
$27 = 3^3$
$9 = 3^2$

b) Extend the patterns in part a downward. Write the next 5 rows in the pattern.

c) Explain how this pattern shows that $a^{-n} = \frac{1}{a^n}$ when a is a non-zero rational number and n is a natural number.

24. Evaluate each power without using a calculator.

a) 2^{-2} b) $\left(\frac{2}{3}\right)^{-3}$ c) $\left(\frac{4}{25}\right)^{-\frac{3}{2}}$

25. Kyle wants to have \$1000 in 3 years. He uses this formula to calculate how much he should invest today in a savings account that pays 3.25% compounded annually: $P = 1000(1.0325)^{-3}$
How much should Kyle invest today?

26. A company designs a container with the shape of a triangular prism to hold 500 mL of juice. The bases of the prism are equilateral triangles with side length s centimetres. The height, h centimetres, of the prism is given by the formula:

$h = 2000(3)^{-\frac{1}{2}}s^{-2}$

What is the height of a container with base side length 8.0 cm? Give your answer to the nearest tenth of a centimetre.

27. When musicians play together, they usually tune their instruments so that the note A above middle C has frequency 440 Hz, called the *concert pitch*. A formula for calculating the frequency, F hertz, of a note n semitones above the concert pitch is:

$F = 440(\sqrt[12]{2})^n$

Middle C is 9 semitones below the concert pitch. What is the frequency of middle C? Give your answer to the nearest hertz.

4.6

28. Simplify. Explain your reasoning.

a) $(3m^4n)^2$ b) $\left(\dfrac{x^2y}{y^{-2}}\right)^{-2}$

c) $(16a^2b^6)^{-\frac{1}{2}}$ d) $\left(\dfrac{r^3s^{-1}}{s^{-2}r^{-2}}\right)^{-\frac{2}{3}}$

29. Simplify. Show your work.

a) $(a^3b)(a^{-1}b^4)$ b) $\left(x^{\frac{1}{2}}y\right)\left(x^{\frac{3}{2}}y^{-2}\right)$

c) $\dfrac{a^3}{a^5} \cdot a^{-3}$ d) $\dfrac{x^2y}{x^{\frac{1}{2}}y^{-2}}$

30. Evaluate.

a) $\left(\dfrac{3}{2}\right)^{\frac{3}{2}} \cdot \left(\dfrac{3}{2}\right)^{\frac{1}{2}}$ b) $\dfrac{(-5.5)^{\frac{2}{3}}}{(-5.5)^{-\frac{4}{3}}}$

c) $\left[\left(-\dfrac{12}{5}\right)^{\frac{1}{3}}\right]^6$ d) $\dfrac{0.16^{\frac{3}{4}}}{0.16^{\frac{1}{4}}}$

31. A sphere has volume 1100 cm^3. Explain how to use exponents or radicals to estimate the radius of the sphere.

32. Identify any errors in each solution, then write a correct solution.

a) $\left(s^{-1}t^{\frac{1}{3}}\right)(s^4t^3) = s^{-1} \cdot s^4 \cdot t^{\frac{1}{3}} \cdot t^3$

$= s^{-4}t$

b) $\left(\dfrac{4c^{\frac{1}{3}}}{d^3}\right)^{-3} = \dfrac{-12c^{-1}}{d^0}$

$= -12c^{-1}$

$= \dfrac{1}{12c}$

THE WORLD OF MATH

Historical Moment: The Golden Ratio

The ratio, $\dfrac{1+\sqrt{5}}{2}$: 1, is called the *golden ratio*.

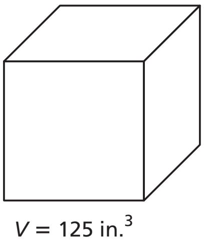
Buildings and pictures with dimensions in this ratio are often considered visually pleasing and "natural." The Greek sculptor Phidias used the golden ratio for the dimensions of his sculptures. His 42-ft. high statue of the Greek god Zeus in the temple in Olympia, created in about 435 B.C.E., was one of the Seven Wonders of the Ancient World. The number $\dfrac{1+\sqrt{5}}{2}$ is often called "phi" after the first Greek letter in "Phidias."

PRACTICE TEST

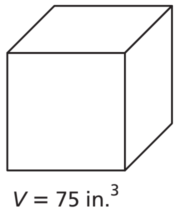
For questions 1 and 2, choose the correct answer: A, B, C, or D

1. The volume V cubic inches of each cube is given. For which cube is the edge length an irrational number?

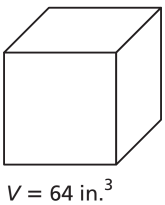
A.



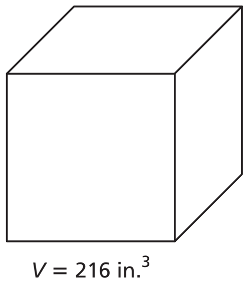
B.



C.



D.



2. Which number is rational?

A. $\sqrt{0.09}$ **B.** $\sqrt{50}$ **C.** $\sqrt[3]{-\frac{64}{121}}$ **D.** π

3. **a)** Which is greater, $\sqrt{70}$ or $5\sqrt{3}$? Justify your answer.
b) Sketch a number line to illustrate the numbers in part a.

4. Evaluate without using a calculator.

a) $\sqrt[4]{\frac{256}{81}}$ **b)** $(-4)^{-2}$ **c)** $0.81^{\frac{3}{2}}$ **d)** $16^{-\frac{1}{2}}$

5. Write $44^{\frac{1}{2}}$ as a radical in simplest form.

6. A student simplified $\frac{x^{-1}y^3}{xy^{-2}}$ as follows:

$$\begin{aligned}\frac{x^{-1}y^3}{xy^{-2}} &= x^{-1+1} \cdot y^{3-2}\\ &= x^0y^1\\ &= y\end{aligned}$$

Is the student correct? If not, describe any errors and write a correct solution.

7. Simplify each expression. Write your answers using positive exponents.

a) $(p^{-2}q^{-1})^2\left(pq^{\frac{1}{2}}\right)^2$ **b)** $\left(\frac{c^6d^5}{c^3d^4}\right)^{-\frac{1}{3}}$

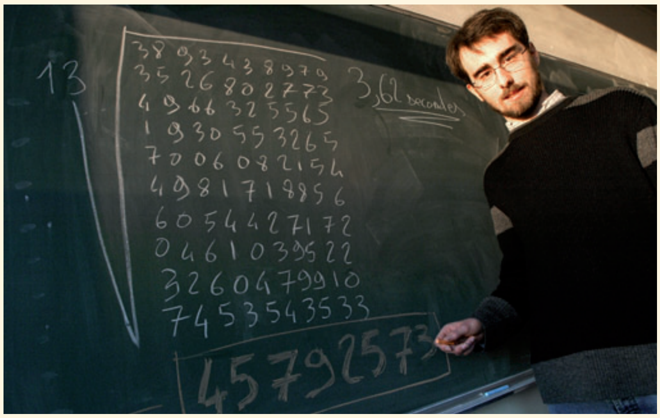
8. Scientists use the formula $d = 0.099m^{\frac{9}{10}}$ to calculate the volume of water, d litres, that a mammal with mass m kilograms should drink in 1 day. Calculate how much water a 550-kg moose should drink in one day.

Human Calculators

Throughout history, there have been men and women so proficient in calculating mentally that they have been called "human calculators."

In 1980, Shakuntala Devi mentally multiplied the numbers 7 686 369 774 870 and 2 465 099 745 779 and gave the correct answer 18 947 668 177 995 426 462 773 730 in 28 s.

In 2004, Alexis Lemaire found the 13th root of a 100-digit number in less than 4 s. In 2007, he was able to find the 13th root of a 200-digit number in a little over 1 min.



PART A: CALCULATING MENTALLY

To calculate squares and square roots using mental math, you have to recognize number patterns, know some elementary algebra, and understand number relationships.

- Use a calculator to evaluate 15^2, 25^2, 35^2, 45^2, and 55^2.
  What patterns do you notice?
  How can you use the patterns to determine the square of a number that ends in 5?
  Use the patterns to evaluate 75^2 and 105^2.
- To square a 2-digit number, you can use the patterns for squaring a binomial:

$$\begin{aligned}41^2 &= (40 + 1)^2\\ &= 40^2 + 2(40)(1) + 1^2\\ &= 1600 + 80 + 1\\ &= 1681\end{aligned}$$

  Adapt this method to determine 39^2 mentally.

- You can use your understanding of number relationships to determine the square roots of 4-digit numbers such as $\sqrt{4489}$ mentally.
  Explain why $60 < \sqrt{4489} < 70$.
  Why must $\sqrt{4489}$ end in 3 or 7?
  What is $\sqrt{4489}$?
  How can the units digit of the radicand help you identify whether the radicand could be a perfect square, and if it is, identify possible roots?

PART B: INVESTIGATING MENTAL CALCULATION METHODS

Invent your own methods or research to find mental math methods that can be used to:

- Square different types of 2-digit numbers.
- Calculate the square root of 4-digit square numbers such as 2601.

PROJECT PRESENTATION

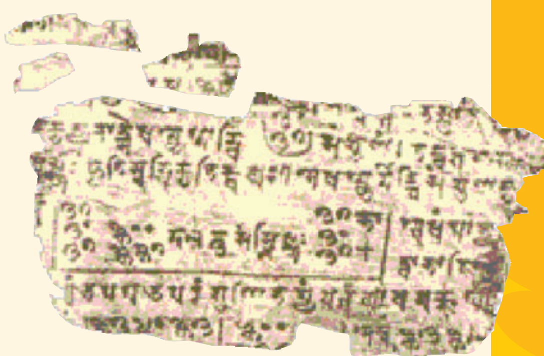
Your completed project can be presented in a written or oral format but should include:

- An explanation of your methods with examples
- An explanation of why the method works; use algebra, number patterns, diagrams, or models such as algebra tiles to support your explanation. You may need to research to find the explanation.

EXTENSION

Because most people do not have extraordinary mental calculation abilities, relatively complex written methods were invented to calculate or estimate roots.

- Use an Internet search or examine older math textbooks to identify some methods or formulas that have been used to calculate or estimate roots, particularly square roots and cube roots. These might include formulas developed by Newton, Heron, and Bakhshali.
- Provide a brief written report with an example of how to use one of these methods. Try to explain why the method works.



The Bakhshali manuscript was found in Pakistan in 1881. It is believed to be a 7th century copy of a manuscript written in the 5th century or earlier.

CUMULATIVE REVIEW Chapters 1–4

1

1. A right rectangular pyramid has base dimensions of 6 m by 4 m and a height of 9 m. Determine the surface area of the pyramid to the nearest square metre.

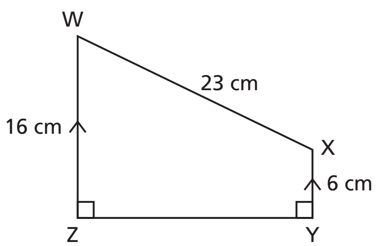
2. A right cone has a slant height of 12 in. and a base diameter of 9 in. Determine the volume of the cone to the nearest cubic inch.

3. The surface area of a sphere is 86 cm^2.
   a) Calculate the diameter of the sphere to the nearest tenth of a centimetre.
   b) What is the radius of the sphere to the nearest inch?

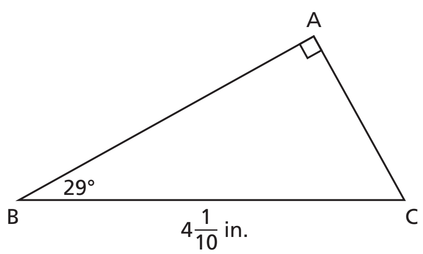
2

4. A tree casts a shadow that is 15 yd. long when the angle between the sun's rays and the ground is 32°. What is the height of the tree to the nearest foot?

5. In this trapezoid, determine the measure of ∠W to the nearest tenth of a degree.



6. a) Determine the perimeter of △ABC to the nearest tenth of an inch.



   b) Determine the area of △ABC to the nearest square inch.

3

7. Determine the greatest common factor and least common multiple of each set of numbers.
   a) 45, 117
   b) 84, 154
   c) 63, 90, 150
   d) 42, 132, 140, 330

8. A cube has volume 50 653 cubic inches. What is its surface area?

9. a) List the first ten perfect squares.
   b) List the first ten perfect cubes.
   c) Which of the perfect cubes in part b are also perfect squares?

10. Factor each polynomial.
    a) $15a^2 - 27a$
    b) $4p + 12p^3 - 6p^2$
    c) $-8d^4 - 14d$
    d) $21w - 28 + 14w^2$
    e) $18x^4y^2 - 4x^3y^3 + 10x^2y^4$
    f) $33n^4p^3 + 11n^2p^2 - 121np^4$

11. Which trinomials can be represented as a rectangle of algebra tiles? What does that tell you about the trinomials?
    a) $t^2 + 8t + 7$
    b) $c^2 + 16c + 4$
    c) $3s^2 + 4s + 6$
    d) $2m^2 + 11m + 15$

12. Expand and simplify. Sketch a rectangle diagram to illustrate each product.
    a) $(d + 5)(d - 3)$
    b) $(5 - s)(9 - s)$
    c) $(-7 + 4g)(7 + 4g)$
    d) $(3k - 7)(2k + 9)$

13. Find an integer to replace □ so that each trinomial can be factored.
    a) $x^2 + \square x + 14$
    b) $b^2 - 5b + \square$
    c) $y^2 + \square y - 18$
    d) $a^2 + 4a + \square$

14. Factor. Check by expanding.
    a) $n^2 + 9n - 22$
    b) $60 - 19m + m^2$
    c) $6r^2 + 23r + 20$
    d) $10n^2 + n - 2$

15. Factor.
a) $3c^2 - 24c - 60$
b) $-5h^2 - 20h + 105$
c) $24c^2 - 87c - 36$
d) $100 - 155a + 60a^2$
e) $4t^2 - 48t + 144$
f) $64 + 8w - 2w^2$
g) $108r^2 - 147s^2$
h) $-70x^2 + 22xy + 12y^2$

16. Expand and simplify.
a) $(2x - 3)(x^2 + 3x - 5)$
b) $(a + 2b)(2a - 5b - 6)$
c) $(4 + t + 3s)(3 - t)$
d) $(n^2 + 2n - 1)(2n^2 - n - 4)$

17. Expand and simplify. Substitute a number for the variable to check each product.
a) $(2c - 5)(c + 6) + (c + 6)(3c - 2)$
b) $(2t - 5)^2 - (2t + 5)(3t - 1)$
c) $(3w + 4)(2w + 7) - (5w + 3)(2w - 6)$
d) $(6d + 3)(2d - 3) - (3d - 4)^2$

18. Factor. Verify by multiplying the factors.
a) $25n^2 + 40n + 16$
b) $24v^2 + 14vw - 3w^2$
c) $81c^2 - 169d^2$
d) $9a^2 - 30ab + 25b^2$

4

19. Use the two consecutive perfect cubes closest to 40 to estimate a value for $\sqrt[3]{40}$. Revise your estimate until its cube is within two decimal places of 40.

20. Locate each number on a number line. Then order the numbers from least to greatest.
$\sqrt[3]{90}, \sqrt{30}, \sqrt[4]{150}, \sqrt[3]{-90}, \sqrt[4]{250}$

21. a) Write each entire radical as a mixed radical.
i) $\sqrt{96}$ ii) $\sqrt[3]{108}$ iii) $\sqrt[4]{144}$
iv) $\sqrt{425}$ v) $\sqrt[3]{648}$ vi) $\sqrt[4]{352}$
b) Write each mixed radical as a entire radical.
i) $5\sqrt{3}$ ii) $2\sqrt[3]{5}$ iii) $11\sqrt[4]{2}$
iv) $3\sqrt{7}$ v) $9\sqrt[3]{4}$ vi) $2\sqrt[5]{3}$

22. a) Write each power as a radical.
i) $50^{\frac{3}{4}}$ ii) $(-2.5)^{\frac{2}{3}}$ iii) $\left(\frac{3}{4}\right)^{1.6}$
b) Write each radical as a power.
i) $\sqrt[3]{8.9^2}$ ii) $\left(\sqrt[4]{\frac{7}{4}}\right)^3$ iii) $\sqrt[5]{(-4.8)^6}$

23. Evaluate each power without using a calculator.
a) $81^{0.75}$ b) $\left(\frac{36}{49}\right)^{\frac{3}{2}}$ c) $(-0.027)^{\frac{5}{3}}$
d) $\left(\frac{4}{9}\right)^{-2}$ e) $16^{-\frac{3}{4}}$ f) $\left(\frac{25}{64}\right)^{-\frac{3}{2}}$
g) $243^{0.6}$ h) $(-0.064)^{-\frac{2}{3}}$ i) $\left(\frac{49}{121}\right)^{-\frac{3}{2}}$

24. Suppose an investor wants to have \$30 000 in 7 years. The interest rate for a savings account is 2.7% compounded annually. The money, P dollars, that she should invest today is given by the formula $P = 30\,000(1.027)^{-7}$. How much money should the investor invest today?

25. Evaluate
a) $\left(\frac{2}{5}\right)^{1.5}\left(\frac{2}{5}\right)^{0.5}$ b) $\dfrac{0.25^{-\frac{2}{3}}}{0.25^{-\frac{5}{3}}}$
c) $\dfrac{\left(0.36^{\frac{5}{2}}\right)\left(0.36^{\frac{3}{2}}\right)}{0.36^{\frac{9}{2}}}$ d) $\dfrac{\left(-\frac{1}{8}\right)^{\frac{7}{3}}\left(-\frac{1}{8}\right)^{\frac{2}{3}}}{\left(-\frac{1}{8}\right)^{\frac{5}{3}}\left(-\frac{1}{8}\right)}$

26. Simplify.
a) $\dfrac{(a^{-2}b^{-1})^{-3}}{a^3b}$ b) $\left(\dfrac{2x^{-4}y^{-3}}{4x^2y^{-5}}\right)^{-4}$
c) $\dfrac{-15a^{-\frac{1}{2}}b}{5ab^{-\frac{3}{2}}}$ d) $\left(\dfrac{x^6z^{-\frac{1}{3}}}{-125y^{-9}z^{\frac{8}{3}}}\right)^{-\frac{1}{3}}$